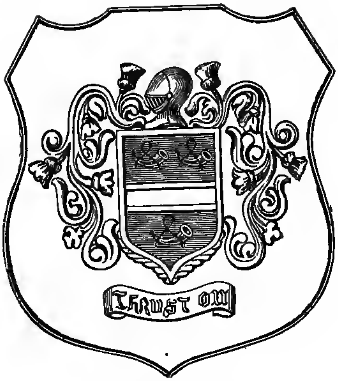

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THEORETICAL MECHANICS

AN

ELEMENTARY TREATISE

ON

THEORETICAL MECHANICS

BY

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PART I: KINEMATICS

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PREFACE.

THE present work owes its existence mainly to the difficulty of finding a good modern text-book suited to the requirements of the American student.

In England it is customary to take a thorough course in elementary mechanics (comprising plane statics and kinetics of a particle) before entering upon the study of higher mathematics; and there is no lack of works of this character (Loney, Macgregor, Selby, Thomson and Tait's Elements, Hicks, Robinson, Browne, Blaikie, Parkinson, Wormell, Lodge, Laverty, etc.), some of which are very well adapted to the purpose. A good course in analytic geometry and the differential and integral calculus will then prepare the student for reading the more advanced English works on analytical statics (Todhunter, Minchin, Routh) and rigid dynamics (Williamson and Tarleton, Routh, Thomson and Tait, Price, Besant, etc.). A similar arrangement is presupposed by most of the French and German treatises.

In many American colleges and universities, however, the student takes up the study of mechanics at a later stage, after having acquired a knowledge of the elements of higher mathematics. A somewhat different treatment of the subject of mechanics is required in this case.

The present volume, which is devoted to kinematics, forms the first of three parts of nearly equal extent. The second part, after an introduction to dynamics in general, takes up statics; it will appear in the fall of this year. The third part, which will be ready in the fall of 1894, is devoted to kinetics.

While the work is intended, first of all, as an introduction to the science of theoretical mechanics as such, the author has constantly kept in mind the particular wants of engineering students, aiming to make it serve as a preparation for the practical applications of this science, and to bring out the utility and importance of the purely mathematical training. General theories are illustrated by special problems and applications in the text, and sets of exercises are inserted to be worked out by the student.

To keep the whole work within reasonable bounds, the more advanced parts of the subject had to be strictly excluded. Bibliographical references have therefore been given for the use of any who are desirous to pursue the subject farther. In accordance with the elementary character of the work, these references are not to original memoirs, but to such standard treatises as can be expected to be found in a well-assorted college library.

At a first reading, the Articles 57–87, 181–214, 221–244, 272–305, can be omitted, also some of the applications and the more difficult exercises.

ALEXANDER ZIWET.

ANN ARBOR, MICH.,
July, 1893.

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THEORETICAL MECHANICS.

INTRODUCTION.

1. The science of theoretical mechanics has for its object the mathematical study of motion.

2. The idea of motion is intimately related to the fundamental ideas of **space, time,** and **mass.** It will be convenient to introduce these consecutively. Thus we shall begin with a purely geometrical study of motion, without regard to the time consumed in the motion and to the mass of the thing moved, the moving object being considered as a mere geometrical configuration. This introductory branch of mechanics may be called the **geometry of motion.**

3. The introduction of the idea of time will then lead us to study the velocity and acceleration of geometrical configurations. This constitutes the subject-matter of **Kinematics** proper. The name Kinematics is, however, used by many authors in a less restricted sense, so as to include the geometry of motion.

4. Finally, endowing our geometrical points, lines, and other configurations with mass, we are led to the ideas of momentum, force, energy, etc. This part of our subject, the most comprehensive of all, has been called **Dynamics,** owing to the importance of the idea of force in its investigation. For the sake of convenience it is usually divided into two branches, **Statics** and

Kinetics. In statics those cases are considered in which no change of motion is produced by the acting forces, or, as it is commonly expressed, in which the forces are in equilibrium. The investigations of statics are therefore independent of the element of time. Kinetics treats in the most general way of the changes of motion produced by forces.

CHAPTER I.

GEOMETRY OF MOTION.

I. *Linear Motion; Translation and Rotation.*

5. Motion consists in change of position.

6. We begin with the simple case of a point moving in a straight line. The position of a point P in a line is determined by its distance $OP=x$ from some fixed point or origin, O, assumed in the line, the length x being taken with the proper sign to express the *sense* (say forward or backward, to the right or to the left) in which it is to be measured on the line. This sense is also indicated by the order of the letters, so that $PO=-OP$, and $OP+PO=0$.

The position of a point in a line is thus fully determined by a single algebraical quantity or co-ordinate; viz. by its abscissa $x=OP$.

7. Let the point P move in the line from any initial position P_0 (Fig. 1) to any other position P_1, and let $OP_0=x_0$, $OP_1=x_1$.

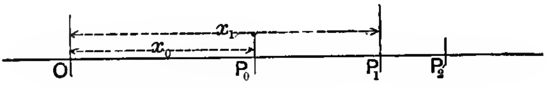

Fig. 1.

This change of position, or *displacement*, is fully determined by the distance $P_0P_1=x_1-x_0$ traversed by the point.

Now let this displacement P_0P_1 be followed by another displacement in the same line, from P_1 to P_2, in the same sense as the former, or in the opposite sense. In either case the total, or **resultant**, displacement is the algebraic sum of the two displacements P_0P_1, P_1P_2, which are called its **components**; *i.e.* we have $P_0P_2 = P_0P_1 + P_1P_2$, or $P_0P_1 + P_1P_2 + P_2P_0 = 0$, whatever may be the positions of the points P_0, P_1, P_2 in the line.

This reasoning is easily extended to any number of component displacements; that is, *the resultant of any number of consecutive displacements of a point in a line is a single displacement equal to the algebraic sum of the components.*

Similar considerations apply to the motion of a point in a curved line provided the displacements be always measured along the curve.

8. Let us next consider the motion of a rigid body. The term **rigid body**, or simply **body**, is used in mechanics to denote a figure of invariable size or shape, or an aggregate of points whose distances from each other remain unchanged. Examples are: a segment of a straight line, a triangle, a cube, an ellipsoid, etc.

Imagine such a body M brought in any manner from some initial position M_0 into any other position M_1. This displacement M_0M_1 is determined by the displacements of the various points of the body. We shall see that, even in the most general case, the displacements of three points of the body determine those of all other points, and consequently the displacement of the whole body.

There are, however, two special cases of motion, translation and rotation, in which the displacement of the body is fully determined by the displacement of a single point: such motions can be called **linear**. There is also a class of motions determined by the displacements of only two points of the body: this is called **plane** motion.

9. *The displacement of a rigid body is called a* **translation** *when the displacements of all of its points are parallel and equal.* It is evident that in this case the displacement of any one point of the body fully represents the displacement of the whole body. The translation M_0M_1 of a rigid body M from a position M_0 to a position M_1 is therefore measured by the rectilinear segment P_0P_1 that represents the displacement of any point P of the body M.

Two or more consecutive translations of a rigid body in the same direction produce a resultant displacement of translation equal to the algebraic sum of the components.

10. When a rigid body has two of its points fixed, the only motion it can have is a rotation about the line joining the fixed points as axis. *In a motion of* **rotation** *all points of the body excepting those on the axis describe arcs of circles whose centres lie on the axis while the points on the axis are at rest.*

The different positions of a rotating body may be referred to any fixed plane passing through the axis of rotation. Any plane of the body passing likewise through the axis will make with the fixed plane an angle θ which varies in the course of the motion. This angle, taken with the proper sign, fully determines the positions of the body.

Let the body rotate from a position θ_0 to a position θ_1; the angle $\theta_1 - \theta_0$ measures the corresponding displacement, or the rotation, just as (Art. 7) the distance $P_0P_1 = x_1 - x_0$ measures the displacement of a point, and hence (Art. 9) the translation of a rigid body.

Two or more consecutive rotations of a rigid body about the same axis give a resultant rotation whose angle is the algebraic sum of the angles of the component rotations.

11. The particular case when the rigid body is a plane figure whose motion is confined to its plane deserves special mention. If one point of such a figure be fixed, the figure can only have a motion of rotation, every other point of the figure describing an arc of a circle whose centre is the fixed point. This point is therefore called the **centre of rotation.** The positions of the figure are given by the angle that any line of the figure passing through the centre makes with any fixed line through the centre in the plane.

12. We have seen that a translation as well as a rotation is measured by a single algebraical quantity, the translation by a distance, the rotation by an angle. This is the reason why such motions may be called *linear* or of one dimension. The two fundamental forms of motion, translation and rotation, are thus seen to correspond to the two fundamental magnitudes of metrical geometry, viz. distance and angle.

It is to be noticed that both for translations in the same direction and for rotations about the same axis the resultant displacement is found by algebraic addition of the components, not only when the components are *consecutive* motions, but even when they are *simultaneous*. Thus we may imagine a point P displaced by the amount P_1P_2 along a straight line while this line itself is moved along in its own direction by an amount Q_1Q_2. The resultant displacement of P is the algebraic sum $P_1P_2+Q_1Q_2$.

13. Translations being measured by distances or lengths, and rotations by angles, we need in mechanics a unit of length and a unit of angle.

The two most important systems of measurement are the C. G. S. (*i.e.* centimetre-gramme-second) system, and the F. P. S. (*i.e.* foot-pound-second) system. The former is frequently called the scientific system; it is based on the international or metric system of weights and measures. The F. P. S., or British system, is still used in England and the United States almost universally in engineering practice.*

14. The unit of length in the C. G. S. system is the **centimetre** (cm.), *i.e.* $\frac{1}{100}$ of the metre. The original standard metre is a platinum bar preserved in the *Palais des Archives* in Paris, a legalized copy of which has been deposited at Washington, D.C. The metre can be defined as the distance between two marks on the standard metre when at a temperature of 0° C.

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* For fuller information on all questions relating to standards and units see J. D. Everett, *Illustrations of the C. G. S. system of units with tables of physical constants;* London, Macmillan, 1891.

In the F. P. S. system, the unit of length is the **foot**, *i.e.* $\frac{1}{3}$ of the standard yard. The original British standard yard is a bronze bar preserved in London. For the United States the yard is defined as the distance between the twenty-seventh and sixty-third divisions of the brass standard yard kept in the Bureau of Weights and Measures at Washington, when the bar is at a temperature of $16\frac{2}{3}°$ C. or 62° F.

The relation between these two fundamental units of length is, according to the *United States Coast and Geodetic Survey Bulletin* No. 9, 1889,

$$1 \text{ cm.} = 0.032\,808\,2 \text{ ft.}$$

For practical use we have the following relations:

$$1 \text{ m.} = 3.2809 \text{ ft.}, \qquad 1 \text{ ft.} = 30.48 \text{ cm.},$$

$$1 \text{ cm.} = 0.3937 \text{ in.}, \qquad 1 \text{ in.} = 2.54 \text{ cm.}$$

15. The unit of angle is either the **degree**, *i.e.* $\frac{1}{360}$ of one revolution, or the **radian**, *i.e.* the angle measured by an arc whose length is equal to the radius.

If α be any angle expressed in radians, and α°, α', α'' the same angle expressed respectively in degrees, minutes, seconds, we have the relations

$$\alpha = \frac{\pi}{180} \cdot \alpha^\circ = \frac{\pi}{10800} \cdot \alpha' = \frac{\pi}{648000} \cdot \alpha'',$$

or $$\alpha = 0.017\,453\,\alpha^\circ = 0.000\,291\,\alpha' = 0.000\,004\,85\,\alpha''.$$

II. *Plane Motion.*

16. The position of a plane figure in its plane is fully determined by the positions of any two of its points since every other point of the figure forms with these two points an invariable triangle. But the position of the figure can of course be determined in other ways; for instance, by the position of one point and that of a line of the figure passing through the point; or by the position of two lines of the figure.

17. Let us now consider the motion of a plane figure F in its plane from any initial position F_0 to any other position F_1. The displacement F_0F_1 can be brought about in various ways. Thus, it would be sufficient to bring any two points A, B (Fig. 2) of the figure F from their initial positions A_0, B_0 in F_0 to their final positions A_1, B_1 in F_1. This can be done, for instance, by first giving the whole figure a translation through a distance A_0A_1 and then a rotation by an angle equal to the angle between A_0B_0 and A_1B_1; or by such a rotation followed by the translation.

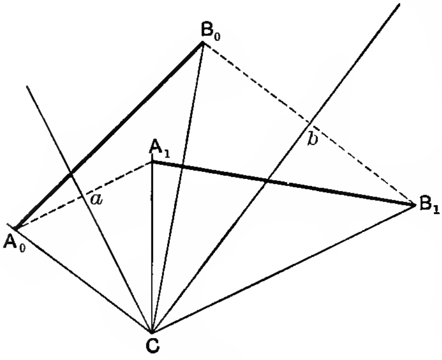

Fig. 2.

Instead of A we might have selected any other point of the figure. But it is important to notice that the angle of rotation required for a given displacement F_0F_1 is always the same, while the translation will differ according to the point selected as centre.

18. This leads us to inquire whether the centre of rotation cannot be so selected as to reduce the translation to zero. Now any rotation that is to bring A from A_0 to A_1 must have its centre on the perpendicular bisector of A_0A_1; similarly for B. Hence the intersection C of the perpendicular bisectors of A_0A_1 and B_0B_1 is the only point by rotation about which both A and B can be brought from their initial to their final positions. That they actually are so brought follows at once from the equality of the angles A_0CB_0 and A_1CB_1 (and hence of the angles A_0CA_1 and B_0CB_1) which are homologous angles in the equal triangles A_0CB_0 and A_1CB_1.

We thus have the proposition: *Any displacement of an invariable plane figure in its plane can be brought about by a single rotation about a certain point* which we may call the *centre* of the displacement.

19. The construction of the centre C given in the preceding article becomes impossible when the bisectors coincide (Fig. 3) and when they are parallel (Fig. 4). In the former case, C is readily found as the intersection of A_0B_0 and A_1B_1. In the latter, *i.e.* whenever $A_0A_1 = B_0B_1$, the centre lies at infinity, and the rotation becomes a translation.

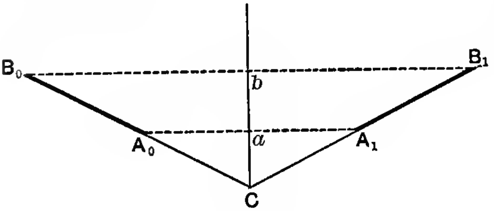

Fig. 3.

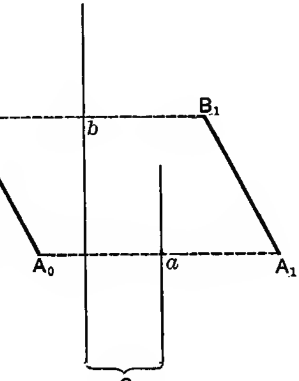

Fig. 4.

Any translation may therefore be regarded as a rotation about a centre at infinity.

20. Let the figure F pass through a series of displacements F_0F_1, F_1F_2, ... $F_{n-1}F_n$. Each displacement has its angle and its centre. If the successive positions F_0, F_1, ... F_n of the figure are taken each very near the preceding one, the angles of rotation will be very small, and the successive centres C_1, C_2, ... C_n will follow each other very closely. In the limit, *i.e.* when for the series of finite displacements we substitute a continuous motion of the figure, the centres C will form a continuous curve (c) and the angles become the infinitely small angles between the successive normals to the paths described by the points of the figure. The point C about which the figure rotates in any one of its positions during the motion is now called the **instantaneous centre**; the locus of the centres, that is the curve (c), is called the **centrode**, or path of the centre. It is apparent that in any position of the moving figure *the normals to the paths of all its points must pass through the instantaneous centre*, and *the direction of motion of any such point is therefore at right angles to the line joining it to the centre.*

21. The centres C are points of the fixed plane in which the motion of the figure F takes place. But in any position F_1 of this figure some point C'_1 of F will coincide with the point C_1 of the fixed plane. Thus, in the case of finite displacements (Fig. 5), let the figure F begin its motion with a rotation of angle θ_1 about a point C_1 of the fixed plane; let C'_1 be the point of the moving figure that coincides during this rotation with C_1.

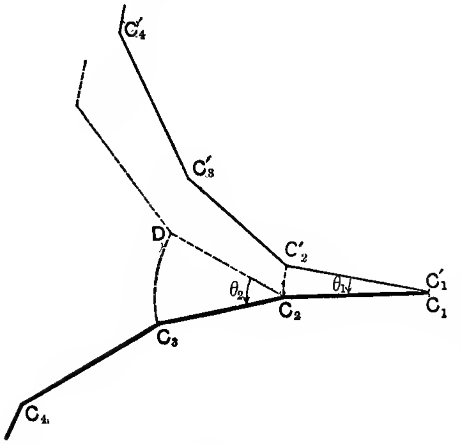

Fig. 5.

The next rotation, of angle θ_2, takes place about a point C_2 of the fixed plane. The point of the moving figure that now coincides with C_2 was brought into the position C_2 by the preceding rotation. Its original position is therefore obtained by turning C_1C_2 back by an angle $-\theta_1$ into the position $C_1C'_2$. The rotation of angle θ_2 about C_2 brings a new point C'_3 of the moving figure to coincidence with the fixed centre C_3; and the original position C'_3 of this point can be determined by first turning C_2C_3 back about C_2 by an angle $-\theta_2$ into the position C_2D, and then turning the broken line C_1C_2D by a rotation of angle $-\theta_1$ about C_1 back into the position $C'_1C'_2C'_3$. Continuing this process we obtain, besides the broken line $C_1C_2C_3\ldots$ formed by joining the successive centres of rotation in the fixed plane, a broken line $C'_1C'_2C'_3\ldots$ in the moving figure formed by joining those points of this figure which in the course of the motion come to coincide with the fixed centres. The whole motion may be regarded as a kind of rolling of the broken line $C'_1C'_2C'_3\ldots$ over the broken line $C_1C_2C_3\ldots$.

22. In the case of continuous motion each of the broken lines becomes a curve, and we have actual rolling of the curve (c'), or **body centrode**, over the curve (c), or **space centrode**. *The continuous motion of an invariable plane figure in its plane may therefore always be produced by the rolling* (without sliding) *of the body centrode over the space centrode.* The point of contact of the two curves is of course the instantaneous centre.

23. It appears from the preceding articles that the continuous motion of a plane figure in its plane is fully determined if we know the centre of rotation for every position of the figure. This centre can be found as the intersection of the normals of the paths of any two points of the figure, so that the motion of the figure will be known if the paths of any two of its points are given. This, however, is only one out of many ways of determining plane motion by two conditions.

Thus the motion may be determined by the condition that a curve of the moving figure should remain in contact with two fixed curves. In this case the instantaneous centre is found as the intersection of the common normals at the points of contact.

The condition that a curve of the moving figure should always pass through a fixed point may be regarded as a special case of the condition just mentioned, one of the fixed curves being reduced to a point.

24. Any curve of the moving figure forms during the motion an **envelope,** the points of the envelope being the intersections of the successive infinitely near positions of the moving curve. Let l, l' be two such successive positions of the curve, A their intersection, C the instantaneous centre; then CA is perpendicular to l as well as to l', and hence to the envelope. The envelope can therefore be constructed by letting fall normals from the instantaneous centres on the corresponding positions of the generating curve.

25. The following examples will illustrate the method of finding the centrodes and the path of any point of the moving figure in plane motion.

Elliptic motion: *Two points of a plane figure move along two fixed lines that are at right angles to each other.*

Let A, B (Fig. 6) be the points moving on the lines Ox, Oy; the perpendiculars to these lines erected at A and B intersect at the instantaneous centre C. Denoting by $2a$ the invariable distance of A and B, we have $OC = AB = 2a$ for all positions of the moving figure. The fixed centrode (c) is therefore a circle of radius $2a$ described about the intersection O of the fixed lines.

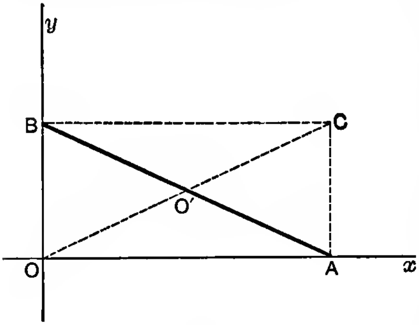

Fig. 6.

To find the body centrode (c') we must construct the triangle ABC for all possible positions of AB. As BCA is always a right angle, the body centrode will be a circle described on AB as diameter. Hence the whole motion can be produced by the rolling of a circle of radius a within a circle of radius $2a$.

The student is advised to carefully carry out the constructions indicated in this as well as the following problems. Thus, in the present case, draw the moving figure, *i.e.* the line AB, in a number of its successive positions in each of the four quadrants, and construct the instantaneous centre C in every case. This gives a number of points of the space centrode. Then take any one position of AB and transfer to it as base all the triangles ABC previously constructed. The vertices of these triangles all lie on the body centrode.

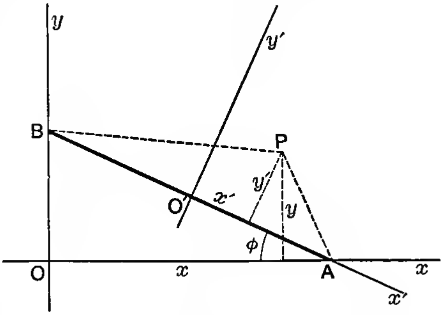

Fig. 7.

26. To find the equation of the path of any point P of the moving figure, let this point be referred to a co-ordinate system fixed in, and moving with, the figure (Fig. 7); let the middle point O' of AB be the origin, and $O'A$ the axis $O'x'$, of this system. Then the co-ordinates x', y' of P in this moving system are connected with its co-ordinates x, y in the fixed system Ox, Oy by the following equations,

$$x = (a + x')\cos\phi + y'\sin\phi,$$
$$y = (a - x')\sin\phi + y'\cos\phi,$$

where ϕ is the angle OAB that determines the instantaneous position of AB. Solving these equations for $\sin\phi$ and $\cos\phi$, squaring and adding, we find for the equation of the path of P

$$\left(\frac{y'x - (a + x')y}{x'^2 + y'^2 - a^2}\right)^2 + \left(\frac{y'y - (a - x')x}{x'^2 + y'^2 - a^2}\right)^2 = 1,$$

or $[(a - x')^2 + y'^2]x^2 - 4ay'xy + [(a + x')^2 + y'^2]y^2 = (x'^2 + y'^2 - a^2)^2,$

which represents an ellipse, since the determinant

$$\begin{vmatrix} (a-x')^2+y'^2, & -2ay' \\ -2ay', & (a+x')^2+y'^2 \end{vmatrix}$$

$$=(x'^2+y'^2+a^2)^2-4a^2(x'^2+y'^2)=(x'^2+y'^2-a^2)^2$$

is necessarily positive.

In general, therefore, the points of the figure describe ellipses; O' describes a circle; A and B describe straight lines, and so does every point on the circle of diameter AB. It is this fact that by rolling a circle within a circle of double diameter the points of the smaller circle are made to describe segments of straight lines, which makes this form of motion of practical importance: it may serve to transform circular into rectilinear motion.

27. Elliptic Motion (continued): *Two points* A, B *of a plane figure move along two fixed lines inclined to each other at an angle* ω (Fig. 8).

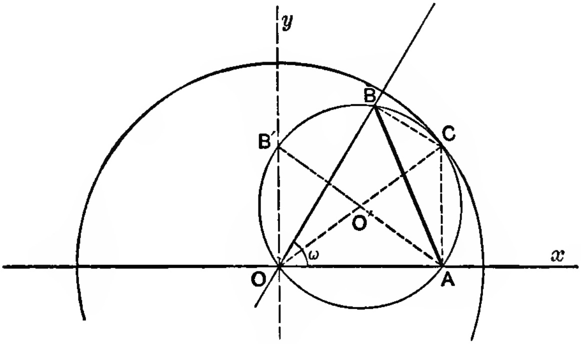

Fig. 8.

This case is readily reduced to the preceding one. The instantaneous centre is found as before; its distance OC from the intersection of the fixed lines OA, OB is again constant and $=AB/\sin\omega$; for O, A, C, B all lie on a circle whose centre O' bisects OC; hence

$$\measuredangle AO'B=2\omega, \text{ and } AB=2AO'\sin\omega=OC\sin\omega.$$

The motion is therefore produced by the rolling of this circle of diameter $AB/\sin\omega$ within a circle of twice this diameter described about O; it is not essentially different from the preceding case (Art. 26). This will also be seen if we take OA as axis of x, the perpendicular to it through O as axis of y. This perpendicular Oy intersects the circle OAB in a point B', which is the end of the diameter $AO'B'$ and moves along Oy during the motion. The points A, B' of the figure move, therefore, along the rectangular lines Ox, Oy, just as in the problem of Art. 26.

28. Connecting Rod Motion: *One point* A *of the figure describes a circle, while another point* B *moves on a straight line, passing through the centre* O *of the circle* (Fig. 9).

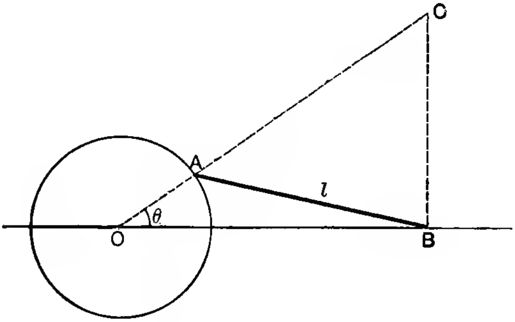

Fig. 9.

With OB as polar axis, the equation of the fixed centrode is

$$r^2\cos^2\theta-2ar\cos^2\theta+a^2=l^2.$$

This, as well as the equation of the body centrode, is of the sixth degree in rectangular Cartesian co-ordinates. But the graphical construction presents no difficulties.

29. Conchoidal Motion: *A point* A *of the figure moves along a fixed straight line* l, *while a line of the figure,* l′, *containing the point* A *always passes through a fixed point* B (Fig. 10).

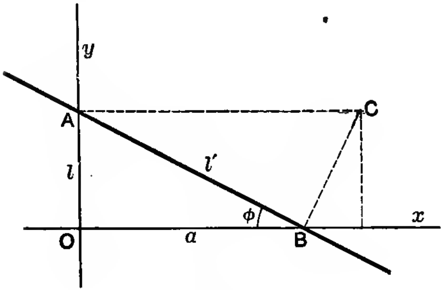

Fig. 10.

The fixed point *B* may be regarded as a circle of infinitely small radius, which the line l' is to touch. The instantaneous centre is therefore the intersection *C* of the perpendiculars erected at *A* on *l* and at *B* on l'.

The fixed centrode is a parabola whose vertex is *B*. To prove this we take the fixed line *l* as axis of *y*, the perpendicular *OB* to it drawn through the fixed point *B* as axis of *x*. Then, putting $\measuredangle OBA=\phi$ and $OB=a$, we have for the co-ordinates of *C*

$$x=a+y\tan\phi,$$
$$y=a\tan\phi;$$

hence $x-a=y^2/a$, or, for *B* as origin and parallel axes, $y^2=ax$.

The equation of the body centrode, for *OB*, *OA* as axes of *x* and *y*, is $a^2(x^2+y^2)=x^4$, or $r\cos^2\theta=a$.

The points of *l* can easily be shown to describe conchoids, whence the name of this form of plane motion.

30. The results obtained in the preceding articles for the motion of a plane figure in its plane apply directly to the motion of a rigid body, if any one point of the body describes a plane curve while a line of the body remains parallel to itself. For in this case all points of the body move in parallel planes, and the motion in any one of these planes determines the motion of the whole figure.

The only modifications required would be that instead of an instantaneous centre we should have an **instantaneous axis**, viz.:

the perpendicular to the plane of motion of any point through the centre of motion of this point; and that the centrodes are now not curves, but cylindrical surfaces rolling one upon the other.

31. Exercises.

(1) Show how to find the direction of motion of any point P rigidly connected with the connecting rod of a steam engine.

(2) A wheel rolls on a straight track; find the direction of motion of any point on its rim. What are the centrodes in this case?

(3) Show how to construct the normal at any point of a conchoid.

(4) Find the equation of the fixed centrode when a line l' of a plane figure always touches a fixed circle O, while a point A of l' moves along a fixed line l.

(5) Show that, in (4), the fixed centrode is a parabola when the fixed circle touches the fixed line.

(6) Two straight lines l', l'' of a plane figure constantly pass each through a fixed point O', O''; investigate the motion.

(7) Four straight rods are jointed so as to form a plane quadrilateral $ABDE$ with invariable sides and variable angles. One side AB being fixed, investigate the motion of the opposite side; construct the centrodes graphically.

(8) Let a straight line l in a fixed plane be brought by a finite displacement from an initial position l_0 into a final position l_1; and let P be any point of l, P_0 its initial position (in l_0), P_1 its final position (in l_1). Then the following propositions can be proved:

(*a*) The middle points of the displacements P_0P_1 of all points P of l lie in a straight line;

(*b*) the lines P_0P_1 envelop a parabola;

(*c*) the projections of the displacements P_0P_1 on the line joining their middle points are all equal;

(*d*) if l have a continuous motion in the plane, the tangents to the paths of all its points envelop a parabola of which the instantaneous centre is the focus and l the tangent at the vertex.

III. *Spherical Motion.*

32. The motion of a spherical figure of invariable form on its sphere presents a close analogy to plane motion; in fact, plane motion is but a special case of spherical motion, since a plane may be regarded as a sphere of infinite radius.

33. By a generalization similar to that of Art. 31, the study of the motion of a spherical figure on its sphere leads directly to the laws of motion of a rigid body having one fixed point. For the motion of such a body is evidently determined by the spherical motion on any sphere described about the fixed point.

34. Let us consider any two positions F_0 and F_1 of a spherical figure F on its sphere, and let O be the centre of the sphere. Just as in the case of plane motion (Art. 18) the displacement F_0F_1 can always be brought about by a single rotation about a point C on the sphere, or what amounts to the same, by a single rotation about the *axis* OC. The proof is strictly analogous to that given in Art. 18. We first remark that the position of the figure on the sphere is fully determined by the position of two of its points, say A and B (Fig. 11), since any third point forms with these an invariable spherical triangle. Let A_0, B_0 be the positions of A, B in F_0; A_1, B_1 their positions in F_1; and draw the great circles A_0A_1 and B_0B_1. Their perpendicular bisectors intersect in two points C, D which are the ends of a diameter of the sphere. CD is the axis of the displacement F_0F_1, and the angle A_0CA_1, or B_0CB_1, gives the angle of the displacement.

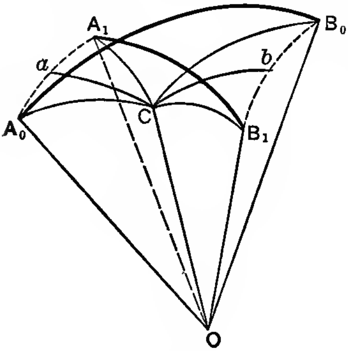

Fig. 11.

35. If we consider a series of positions of the moving figure, $F_0, F_1, F_2, \ldots$, we obtain a series of axes of rotation, say $c_1, c_2, \ldots$; and in the limit when these positions follow one another at infinitely near intervals, the axes $c_1, c_2, \ldots$ will form a cone fixed in space, with the vertex at the centre O of the sphere. The points $C_1, C_2, \ldots$ where these axes intersect the sphere form a curve (c) on the fixed sphere, while the points $C'_1, C'_2, \ldots$ of the moving figure with which these fixed points come to coincide form a spherical curve (c') invariably connected with the moving figure. The whole motion may be produced by the rolling of the curve (c') over the curve (c), or also by the rolling of the corresponding cones one over the other. We have thus the proposition that *any continuous motion of a rigid body having a fixed point can be produced by the rolling of a cone fixed in the body on a fixed cone, the vertices of both cones being at the fixed point.*

IV. *Screw Motion.*

36. The position of a rigid body in space is fully determined by the position of any three of its points not situated in the same straight line. For any fourth point of the body will form an invariable tetrahedron with these three points. As two points determine a straight line, the position of a rigid body may also be given by the position of any one of its lines or by the positions of two intersecting or parallel lines of the body.

37. The position of a point being determined by its three co-ordinates requires three conditions to be fixed. A point is therefore said to have three *degrees of freedom* when its position is not subject to any conditions. One conditional equation between its co-ordinates restricts the point to the surface represented by that equation; the point is then said to have but two degrees of freedom and one *constraint.* Two conditions would restrict the point to a line, the curve of intersection of the two surfaces represented by the equations of condition;

the point has then but one degree of freedom and two constraints.

A rigid body that is perfectly free to move has six degrees of freedom. For we have seen that its position is fully determined when three of its points not in the same line are fixed. The nine co-ordinates of these points are, however, not independent; they are connected by the three equations expressing that the three distances between the three points are invariable. Thus the number of independent conditions is $9-3=6$.

A rigid body with one fixed point has three degrees of freedom and therefore three constraints. For it takes two more points, *i.e.* six co-ordinates, to fix the position of the body; and the distances of these two points from each other and from the fixed point being invariable, there are again three conditional equations to which the six co-ordinates are subject. The three co-ordinates of the fixed point may be regarded as the three constraints.

A rigid body with two fixed points, i.e., *with a fixed axis, has one degree of freedom, and five constraints.* Indeed, the six co-ordinates of the two fixed points are equivalent to five constraining conditions, since the distance of these two points is invariable.*

38. Let us now consider any two positions M_0, M_1 of a rigid body M, given by the positions A_0, B_0, C_0 and A_1, B_1, C_1 of three points A, B, C of the body. The displacement M_0M_1 can be effected in various ways. Thus we might for instance begin by giving the whole body a translation equal to A_0A_1 which would bring the point A to its final position while all other points of the body would be displaced by distances parallel and equal to A_0A_1. As the body has now one of its points, A, in its final position, it will (by Art. 34) require only a single rotation about a certain axis passing through this point to bring the whole body into its final position. It thus appears that any displacement of a rigid body can be effected by subjecting the body first to a translation and then to a rotation (or *vice versa*, as is easily seen); and this can be done in an infinite number of ways, as the displacement of *any* point of the body may be selected for the translation.

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* Interesting remarks on the mechanical means of producing constraints of various degrees will be found in THOMSON and TAIT, *Natural philosophy*, London, Macmillan, new edition, 1879, Art. 195 sq. (Part I., p. 149).

39. It is to be noticed that for all these different ways of effecting the displacement M_0M_1 the *direction* of the axis of rotation and the *angle* of rotation are the same. To see this more clearly, let the displacement be effected first by the translation A_0A_1 and a rotation of angle α about the axis a_1 passing through A_1; and then let the same displacement be produced by the translation B_0B_1 of some other point B and a rotation of angle β about an axis b_1 passing through B_1. We wish to show that a_1 and b_1 are parallel and that the angles α and β are equal.

Consider a plane π of the rigid body which in its original position π_0 is perpendicular to the axis a_1. The translation A_0A_1 transfers it into a parallel position and the rotation α about a_1 turns it in itself into its final position π_1; hence π_0 and π_1 are parallel. The translation B_0B_1 likewise moves π into a position parallel to the original one; and as its final position, π_1, is parallel to π_0, the axis of rotation b_1 must necessarily be perpendicular to π_0 and π_1, that is b_1 must be parallel to a_1.

Again, any straight line l in π remains parallel to its original position l_0 after the translations A_0A_1 and B_0B_1. Its change of direction is due to the rotations alone; the angle of rotation must therefore be the same for both rotations, viz. equal to the angle (l_0l_1) formed by the initial and final positions of the line l.

40. Among the different combinations of a translation with a rotation effecting the displacement M_0M_1 there is one of particular importance; it is that for which the axis of rotation is parallel to the translation.

Let us again consider the plane π perpendicular to the common direction of the axes of rotation. To bring any three points of this plane into their final position it is only necessary to give the body a translation at right angles to π such as to bring π into its final position and then to add the necessary rotation for plane motion.

We have therefore the important proposition that *it is always possible to bring a rigid body* M *from any position* M_0 *into any other position* M_1 *by a translation combined with a rotation about an axis parallel to the direction of translation,* and this can be done in only one way. The axis so determined is called the **central axis** of the displacement.

The order of translation and rotation about the central axis is indifferent; indeed, translation and rotation might take place simultaneously.

41. A motion of a rigid body consisting of a rotation about an axis combined with a translation parallel to the axis is called a *screw motion*, or a *twist*. We have proved therefore, in Art. 40, that *the most general displacement of a rigid body can be brought about by a single twist.*

42. To construct the central axis and find the translation and angle of the twist when the displacement is given by the positions A_0, B_0, C_0 and A_1, B_1, C_1 of three points of the body, we first remark that the projection on the central axis of the displacement of any point, say A_0A_1, is equal to the translation of the twist, and hence the projections of the displacements of all points of the body (such as A_0A_1, B_0B_1, C_0C_1) are all equal. If therefore from any point O we draw lines OA, OB, OC equal and parallel to A_0A_1, B_0B_1, C_0C_1, their ends A, B, C will lie in a plane π perpendicular to the central axis, and the perpendicular p dropped from O on this plane π will represent in length and direction the translation of the twist.

The direction of the central axis being thus determined, we find its position in space by projecting the displacements of any two of the three given points, say A_0A_1 and B_0B_1, on the plane

π, and finding the intersection of the perpendicular bisectors of these projections. This intersection is evidently a point of the central axis, and a perpendicular through it to the plane π will give the central axis in position.

43. In the case of continuous motion there exists a central axis for every position of the body; but its position both in space and in the body in general varies in the course of the motion. The central axis at any moment is therefore called in this case the **instantaneous axis.**

44. The straight lines of space which during the progress of the motion become instantaneous axes for the infinitely small twists of the body form a ruled surface. Similarly, the lines of the moving body which in the course of the motion come to coincide with these axes generate another ruled surface. In any given position of the body these two surfaces are in contact along a line (the instantaneous axis) which is a generator in each of the two surfaces. The body has an infinitely small rotation about this line and at the same time slides along this line through an infinitely small distance.

Thus *the continuous motion of a rigid body in the most general case can be regarded as consisting of the combined rolling and sliding of one ruled surface over another.*

V. *Composition and Resolution of Displacements.*

1. TRANSLATIONS; VECTORS.

45. All the points of a rigid body subjected to a translation describe parallel and equal lines (Art. 9). The translation of the body is therefore fully determined by the displacement A_0A_1 of any one point A of the body (Fig. 12), and can be represented geometrically by A_0A_1 or any line equal and parallel to it, like o1.

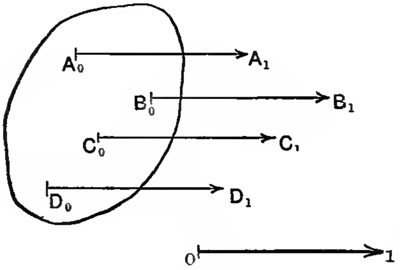

Fig. 12.

A segment of a straight line of definite length, direction, and sense is called a **vector**. The *sense* of the vector (see Art. 6) which expresses whether the translation is to take place from o to 1 or from 1 to o, is indicated graphically by an arrow-head, and in naming the vector, by the order of the letters, o1 and 1o being vectors of opposite sense.

46. Imagine a rigid body subjected to two successive translations. From any point o (Fig. 13) draw a vector o1 representing the first translation, and from its end 1 a vector 12 representing the second translation. The vector o2 will then represent a translation that would bring the body directly from its initial to its final position. This vector o2 is called the **geometric sum**, or the **resultant**, of the vectors o1 and 12, which are called the **components**. The operation of combining the components into a resultant, or of finding the geometric sum of two vectors, is called **geometric addition**, or **composition**, of vectors.

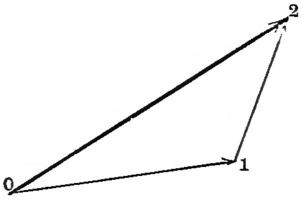

Fig. 13.

47. The process of geometric addition explained in Art. 46 for the case of *two* components is readily extended to the general case of *n* components. It thus appears that *the succession of any number of translations of a rigid body has for its resultant a single translation whose vector is found by geometrically adding the vectors of the component translations.* (Compare Art. 7.)

48. The order in which vectors are combined, or added, is indifferent for the result. This is directly apparent from a figure in the case of two vectors (Fig. 14). For the case of *n* vectors it follows from the consideration that any order of the vectors can be obtained by repeated interchanges of two successive vectors.

Geometric addition agrees, therefore, with algebraic addition in being *commutative.*

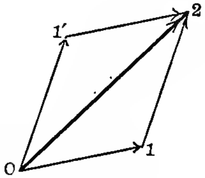

Fig. 14.

49. The vector, as the geometric symbol of a translation, has *length*, *direction*, and *sense;* but it is not restricted to any definite *position*, the same translation being represented by all equal and parallel vectors. We express this by saying that *two vectors are equal if they are of the same length, direction, and sense.*

Translations are not the only magnitudes in mechanics that can be represented by vectors. We shall see later that velocities, accelerations, moments of couples, etc., can all be represented by vectors and are therefore compounded into resultants and resolved into components by geometric addition and subtraction. In this lies the importance of this subject which in its special application to translations might appear too simple and self-evident to require extended presentation.

The case when the vectors represent concurrent forces is probably known to the student from elementary physics as the "parallelogram" or "polygon" of forces.

50. A translation may be *resolved* into two or more translations by resolving its vector into components.

When the resultant translation and one of its components are given by their vectors, the process of finding the other component is called **geometric subtraction.** It is effected, like algebraic subtraction, by reversing the *sense* of the component to be subtracted, and then geometrically adding it to the resultant (Fig. 15). In other words, the *geometric difference* of two vectors AB and CD is found by geometrically adding to AB a vector equal but opposite to CD. Thus, in Fig. 15, 02 is made equal and parallel to AB; 21 is equal and parallel to CD reversed, that is to DC; 01 is the required difference.

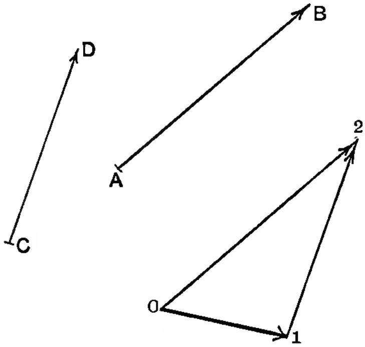

Fig. 15.

51. The composition of translations by geometric addition of their vectors (Art. 47) holds, not for *successive* translations only, but, owing to the commutative law (Art. 48), for *simultaneous* translations as well. This is easily seen by resolving the components into infinitesimal parts.

To obtain a clear idea of two simultaneous translations it is best to imagine the body as having one of these translations with respect to some other body, while the latter itself is subjected to the other translation. A man walking across the deck of a vessel in motion, an object let fall in a moving carriage, a spider running along a branch swayed by the wind, are familiar examples.

52. This leads us to the idea of **relative motion.**

Properly speaking, all motion is relative; that is, we can conceive of the motion of a body only with regard to some other body, called the **body of reference.** If the latter be regarded as fixed, the motion of the former is called its **absolute motion.**

Thus in speaking of the motion of a railway train, we usually regard the earth as fixed and can thus call the displacement of the train from one station to another an *absolute displacement*. If, however, the motion of the earth with regard to the sun be taken into account, the displacement of the train from station to station is the relative displacement of the train with respect to the earth; and its absolute displacement would be found by combining this relative displacement with the absolute displacement of the earth (with respect to the sun regarded as fixed).

53. It follows that when the two displacements are translations *the absolute displacement of the body will be found by geometrically adding its relative displacement to the absolute displacement of the body of reference.* And conversely, *the relative displacement of a body is found by geometrically subtracting from its absolute displacement the absolute displacement of the body of reference.*

54. Analytically, the composition and resolution of vectors is merely a problem of trigonometry. Thus, the resultant of two vectors is the diagonal of the parallelogram formed by the two vectors as adjacent sides; the resultant of three vectors is the diagonal of the parallelepiped having the three vectors as concurrent edges.

55. In the case of more than two or three vectors, however, the solution by ordinary trigonometry would become rather tedious, and it is best to proceed as follows:

Assume an origin O and three rectangular axes Ox, Oy, Oz, and project each vector on the three axes; let X, Y, Z be its projections. These projections X, Y, Z are three vectors whose geometrical sum is equal to the vector. If n vectors were originally given, we should now have them replaced by $3n$ components of which n lie in each axis. The components lying in the same axis can be added algebraically; let their respective sums be ΣX, ΣY, ΣZ. The n vectors are therefore equivalent to the three vectors ΣX, ΣY, ΣZ, which form the concurrent edges of a rectangular parallelepiped whose diagonal drawn through the origin O is the resultant vector $OR = R$, *i.e.*

$$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2 + (\Sigma Z)^2}.$$

The direction of this vector is given by the equations

$$\cos\alpha = \frac{\Sigma X}{R},\quad \cos\beta = \frac{\Sigma Y}{R},\quad \cos\gamma = \frac{\Sigma Z}{R},$$

where α, β, γ are the angles made by OR with the axes Ox, Oy, Oz, respectively.

If all the vectors lie in the same plane, we have simply:

$$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2},\quad \tan\alpha = \frac{\Sigma Y}{\Sigma X}.$$

56. Exercises.

(1) A ship sails first 5 miles N. 30° E., then 12 miles N. 60° E., and finally 25 miles E. 75° S. Find distance and bearing of the point reached: (*a*) graphically, (*b*) analytically.

(2) Is a scale of 8 miles to the inch sufficient to obtain the results of Ex. (1) correctly to whole miles and degrees?

(3) A rigid body undergoes three translations, of 1, 2, and 3 feet, whose directions are respectively parallel to the three sides of an equilateral triangle taken the same way round. Find the resulting displacement.

(4) A ship is carried by the current 2 miles due W., and at the same time by the wind 4 miles due N.E., and by her screw 11 miles E. 30° S. Find her resultant displacement.

(5) A ferry-boat crosses a river in a direction inclined at an angle of 60° to the direction of the current. If the width of the river be half a mile, what are the component displacements of the boat along the river and at right angles to it?

(6) Two vectors of equal length a are inclined to each other at an angle α. Find the resultant in magnitude and direction.

(7) For what angle α, in Ex. (6), is the resultant equal in magnitude: (*a*) to each component a? (*b*) to $\frac{1}{3}a$?

(8) Resolve a vector a into two components making with the vector angles of 30° and 45° on opposite sides.

(9) Steering his boat directly across a river whose current is due west, a man arrives on the opposite bank at a point from which the starting-point bears S.E.; the width of the river being 1200 feet, how far has he rowed? What is the absolute, and what the relative, displacement of the boat?

(10) Assuming a raindrop to fall 50 feet in a second in a vertical direction, find in what direction it appears to be falling to a man: (*a*) walking at the rate of 5 feet per second, (*b*) driving at the rate of 10 feet per second, (*c*) riding on a bicycle at 25 feet per second, (*d*) in a railroad car running 60 feet per second.

(11) Find in magnitude and direction the resultant of 8 translations of 1, 2, 3, 4, 5, 6, 7, 8 feet, respectively, each component making an angle of 45° with the preceding one: (*a*) graphically, (*b*) analytically.

(12) If a, b, c are three vectors whose geometric sum is 0, prove that $a/\sin(bc) = b/\sin(ca) = c/\sin(ab)$.

(13) Find the resultant of two translations represented in magnitude and direction by two rectangular chords of a circle drawn from a point on its circumference.

(14) From a point C in the plane of a circle whose centre is O, draw two lines at right angles to each other so as to intersect the circle in A, A' and B, B', respectively. Show that the resultant of the four vectors CA, CA', CB, CB' is equal to twice CO.

(15) Prove that the geometric sum of two vectors P_0P_1, P_0P_2 issuing from the same point P_0 passes through the middle point G of P_1P_2 and has a length $= 2\,P_0G$.

(16) Prove that the geometric sum of two vectors P_0P_1 and P_0P_2 is equal to $(n+1)P_0G$ if G be found as follows: on P_0P_1 take Q so that $P_0Q = \frac{1}{n} P_0P_1$, and on QP_2 take G so that $QG = \frac{1}{n+1} QP_2$.

(17) Show that Ex. (15) is a special case of Ex. (16).

(18) Prove the following rule for constructing the geometric sum of n vectors P_0P_1, P_0P_2, P_0P_3, $\cdots P_0P_n$ issuing from the same point P_0: on P_1P_2 take G_1 so that $P_1G_1 = \frac{1}{2}P_1P_2$; on G_1P_3 take G_2 so that $G_1G_2 = \frac{1}{3}G_1P_3$; on G_2P_4 take G_3 so that $G_2G_3 = \frac{1}{4}\,G_2P_4$; and so on. If G be the last point so determined, the geometric sum of the n vectors is $= nP_0G$.

2. ROTATIONS; ROTORS.

57. When a rigid body has a motion of rotation about a fixed axis, all its points with the exception of those on the axis describe circular arcs whose centres are situated on the axis (Art. 10).

The elements determining a *rotary displacement*, or a *rotation*, are the axis and the angle of rotation. These elements can be represented by a single geometrical symbol; we have only to lay off on the axis of rotation a length $O1$ (Fig. 16) representing on some scale the magnitude of the angle θ. An arrow-head can be used to mark the *sense* of the angle. It is customary, at least in English works on mechanics, to adopt the counter-clockwise sense of rotation as positive. The arrow-head should then be placed at that end of the line representing the angle θ from which the rotation appears counter-clockwise in a plane through the other end at right angles to the axis. The arrow then points in the direction in which an ordinary screw moves when turned in the positive sense.

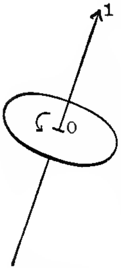

Fig. 16.

This geometrical symbol of a rotation, $O1$, has been called a **rotor.** It becomes of importance in the case of infinitesimal rotations, as we shall see later (Art. 68).

58. Two or more rotations about the same axis can evidently be combined into a single rotation about the same axis whose angle is the algebraic sum of the angles of the component rotations (Art. 12). As regards rotations about different axes, we have to distinguish three cases: intersecting axes, parallel axes, and crossing or skew axes.

It will be shown in the following articles that rotations about intersecting or parallel axes can always be combined into a single rotation which may happen to reduce to a translation.

Rotations about skew axes cannot in general be reduced to a single rotation or translation; it will be shown in the next section (Arts. 74–79) that they reduce to a twist, or screw motion.

59. Intersecting Axes. *The resultant of two successive rotations,* θ_1 *about* l_1 *and* θ_2 *about* l_2, *when the axes* l_1 *and* l_2 *intersect in a point O, is a single rotation of angle* θ *about an axis* l *passing through O.* The trihedral formed by l_1, l_2 and l has at l_1 a dihedral angle $=\frac{1}{2}\theta_1$, at l_2 a dihedral angle $=-\frac{1}{2}\theta_2$, while its exterior angle at l is $=\frac{1}{2}\theta$; that is, we have on a sphere of radius 1 described about O:

$$\cos\tfrac{1}{2}\theta=\cos\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_2-\sin\tfrac{1}{2}\theta_1\sin\tfrac{1}{2}\theta_2\cos(l_1l_2), \tag{1}$$

$$\frac{\sin(l_1l)}{\sin\frac{1}{2}\theta_2}=\frac{\sin(ll_2)}{\sin\frac{1}{2}\theta_1}=\frac{\sin(l_1l_2)}{\sin\frac{1}{2}\theta}. \tag{2}$$

The truth of this proposition will appear by considering Fig. 17. The rotation θ_1 about the axis l_1 brings the axis l_2 into its final position l'_2. The rotation θ_2 about l'_2 brings l_1 into its final position l'_1. The planes bisecting the dihedral angles θ_1 at l_1 and θ_2 at l'_2 intersect in a line l which by the rotation θ_1 about l_1 is brought into the position l', and by the rotation θ_2 about l'_2 is brought back into its original position l. The effect of the two rotations taken in this order is therefore to leave the line l in its place; that is, the resultant of the two successive rotations is a single rotation about l as axis. Moreover, inspection of the figure shows that a rotation about l by an angle equal to twice the exterior angle of the trihedral ll_1l_2 at l brings l_1 and l_2 into their final positions l'_1 and l'_2.

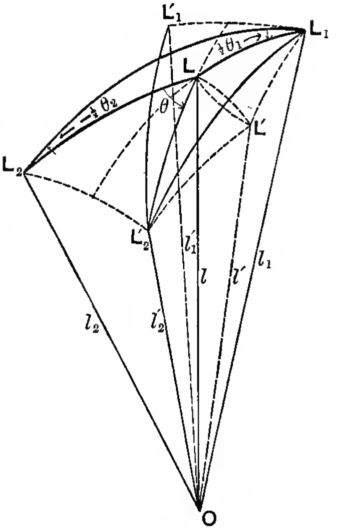

Fig. 17.

60. It is to be noticed that l_1 and l_2 are here regarded as lines of the rigid body; and while l_1 coincides with the position of the first axis of rotation in space, the second axis of rotation in space has the position l'_2, and not l_2. It follows that, *in general. the order of the two rotations is not indifferent.* But by repeating the construction, any number of rotations taken in a definite order can be combined into a single rotation provided every axis intersects the axis of the resultant of all preceding rotations.

61. Again, in finding l from l_1 and l_2, the positions of the axes in the rigid body, as we did in Art. 59, the angle $\frac{1}{2}\theta_1$ is to be applied to the plane $l_1 l_2$ at l_1 in its proper sense, *i.e.* on that side towards which the rotation about l_1 takes place; but $\frac{1}{2}\theta_2$ at l_2 is to be applied to this plane in the opposite sense. If, however, we wish to construct l from the *absolute* positions of the axes of rotation in space, l_1 and l'_2, we have to use $-\frac{1}{2}\theta_1$ and $+\frac{1}{2}\theta_2$.

62. In the case of two *infinitely small rotations*, say $d\theta_1$ and $d\theta_2$, about intersecting axes l_1, l_2, the construction gains remarkable simplicity. The resulting axis l falls into the plane of the given axes.

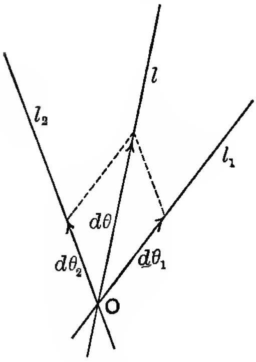

Fig. 18.

Substituting $d\theta$ for $\sin\theta$ and $1-\frac{1}{2}d\theta^2$ for $\cos\theta$, the equations of Art. 59 assume the form

$$d\theta^2 = d\theta_1{}^2 + d\theta_2{}^2 + 2\,d\theta_1 d\theta_2 \cos(l_1 l_2), \qquad (1')$$

$$\frac{\sin(l_1 l)}{d\theta_2} = \frac{\sin(l l_2)}{d\theta_1} = \frac{\sin(l_1 l_2)}{d\vartheta}. \qquad (2')$$

These equations show that $d\theta$ can be found by geometrically adding the rotors (Art. 57) representing the rotations $d\theta_1$ and $d\theta_2$. In other words, the components $d\theta_1$ and $d\theta_2$ (or lengths proportional to them) being laid off on their respective axes (Fig. 18), the resultant rotation

$d\theta$ will be found in magnitude and direction as the diagonal of the parallelogram whose adjacent sides are $d\theta_1$ and $d\theta_2$, just as in the case of translations (Art. 46). The importance of this proposition will appear later (Art. 176).

It is to be noticed that, in the case of infinitesimal rotations, the order of succession in which they take place is obviously indifferent; they can therefore be imagined to take place simultaneously.

63. Parallel Axes. The composition of two successive rotations about parallel axes is not essentially different from the composition of rotations about two intersecting axes. The trihedral ll_1l_2 of Fig. 17, formed by the given axes l_1, l_2, and the resulting axis l, becomes now a triangular prism, and the spherical construction is replaced by a construction in a plane at right angles to the axes. Fig. 19 shows this construction for the case of two rotations having the same sense (θ_1 and θ_2 being of the same sign); Fig. 20 illustrates the case of two opposite rotations. The letters have the same meaning as in Fig. 17.

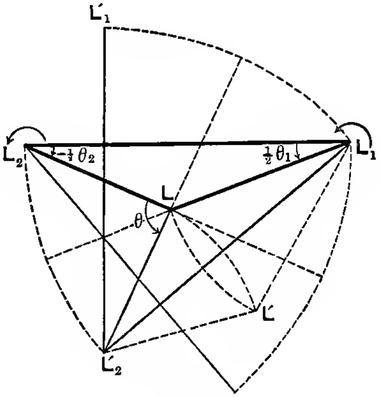

Fig. 19.

The signs of θ_1 and θ_2 being taken into account, the formulæ of Art. 59 are now replaced by the following:

$$\theta = \theta_1 + \theta_2, \qquad (1'')$$

$$\frac{L_1L}{\sin \frac{1}{2}\theta_2} = \frac{LL_2}{\sin \frac{1}{2}\theta_1} = \frac{L_1L_2}{\sin \frac{1}{2}\theta}. \qquad (2'')$$

The order of two finite rotations about parallel axes is not invertible.

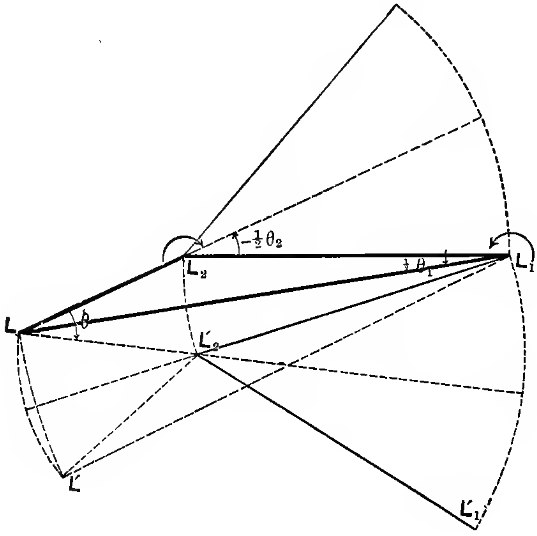

Fig. 20.

By repeating the above construction it is evidently possible to find the resultant of any number of successive rotations about parallel axes, the rotations being taken in a definite order.

64. The particular case of two *equal and opposite* rotations about parallel axes deserves special consideration. The point L lies at infinity; hence, the axis of rotation being at an infinite distance, the resulting motion is a translation (Art. 19). This will also appear from Fig. 21; the first rotation, about l_1, brings the plane l_1l_2 into the position $l_1l'_2$; the following rotation, about l'_2, brings it into the position $l'_2l'_1$ which is parallel to the original position l_2l_1. The whole body has thus been moved parallel to itself in the direction $L_1L'_1$, and the magnitude of this translation is

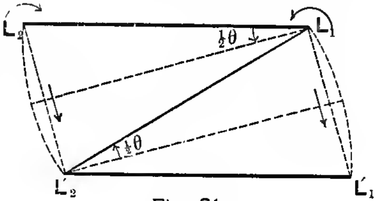

Fig. 21.

$$L_1L'_1 = L_2L'_2 = 2\,L_1L_2 \sin\frac{\theta}{2}, \tag{3}$$

where θ is the angle of rotation about each axis, and L_1L_2 is the distance of the axes.

The order of the rotations is evidently not invertible.

65. We have seen in the preceding article that *two equal and opposite rotations about parallel axes produce a translation at right angles to the axes of rotation.* A translation can therefore always be replaced by two such rotations. It follows that *a translation followed by a rotation about an axis at right angles to the direction of translation can be replaced by a single rotation about a parallel axis.* To find this resulting rotation it is only necessary to replace the translation by two parallel equal and opposite rotations having the same effect (Art. 64); the three rotations so obtained have parallel axes and can therefore (Art. 63) be combined into a single one.

66. The case of two *infinitely small rotations* (Fig. 22) is again of particular importance, as we shall see later on. The formulæ of Arts. 59 and 63 become in this case

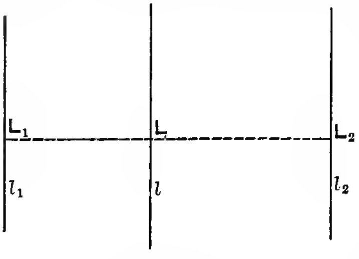

Fig. 22.

$$d\theta = d\theta_1 + d\theta_2, \tag{1'''}$$

$$\frac{L_1L}{d\theta_2} = \frac{LL_2}{d\theta_1} = \frac{L_1L_2}{d\theta}. \tag{2'''}$$

The axis l of the resulting rotation lies therefore in the plane of the given axes l_1, l_2 and divides their distance in the inverse ratio of the angles of rotation. The sense of the segments L_1L, LL_2, L_1L_2 must be taken into account as well as the sense of the angles $d\theta_1$, $d\theta_2$, $d\theta$. The axis l lies between l_1 and l_2 if $d\theta_1$, $d\theta_2$ have the same sense; otherwise it lies outside the space between l_1, l_2 on the side of the axis having the greater angle.

67. Two equal and opposite infinitely small rotations about parallel axes produce an infinitely small translation equal to $L_1L_2 \cdot d\theta$ (see Art. 64, Formula (3)) directed at right angles to the plane of the axes l_1, l_2. Conversely, an infinitely small translation can always be replaced by two equal and opposite infinitesimal rotations.

68. An infinitesimal rotation of angle $d\theta$ about an axis l can be represented (Art. 57) by a rectilinear segment laid off on l equal to $d\theta$, or, to avoid infinitesimal lengths, proportional to $d\theta$. This geometrical symbol of an infinitesimal rotation has all the characteristics of a vector (compare Arts. 45, 49); but it has one more which distinguishes it from the vector representing a translation: it is *localized*, or attached to a definite line; for two equal and parallel rotations about different axes do not represent the same thing. Such a localized vector is called a **rotor.**

69. The theory of rotors is of just as great importance in mechanics as that of vectors (Art. 49). Angular velocities, momenta, forces, all have for their geometrical representatives rotors, *i.e.* rectilinear segments of definite direction, length, sense, and situated on a definite line.

The theory of the composition and resolution of rotors is a matter of pure geometry; it remains the same whatever the rotor may represent. Thus we have seen in Art. 62, in the case of infinitesimal rotations, that concurrent rotors are combined by geometrical addition. The same rule holds for angular velocities, momenta, and forces. In Art. 66 the rule for combining two parallel rotors is explained by the example of infinitesimal rotations. The student acquainted with elementary physics will recognize in this rule the so-called principle of the lever which is based on the composition of parallel forces.

70. Two rotors of equal length and opposite sense situated on parallel lines (Fig. 23) are said to form a **couple.** The two rotors P, P are called the **sides,** their perpendicular distance p the **arm,** and the product Pp the **moment** of the couple.

It has been proved in Art. 67 that a couple of infinitesimal rotations produces an infinitesimal translation. In general, a rotor couple is equivalent to a vector, as we shall see later.

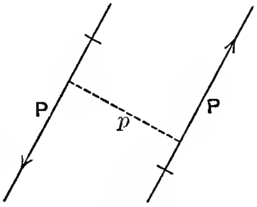

Fig. 23.

71. The converse proposition of Art. 67, viz. that an infinitesimal translation can always be replaced by a couple of infinitesimal rotations, requires a little further consideration.

Suppose we wish to replace the translation ds by a couple. According to Art. 67, the axes l_1, l_2 of the two rotations must be at right angles to ds; the distance L_1L_2 of the axes and the angle of rotation $d\theta$ are only subject to the condition that their product should equal ds, i.e.

$$L_1L_2 \cdot d\theta = ds.$$

There is, therefore, an infinite number of couples equivalent to ds, all having the same moment $L_1L_2 \cdot ds$ and all lying in a plane perpendicular to ds.

It thus appears that the characteristics of a couple are its moment and the aspect of its plane; in other words, a couple (P, p) is equivalent to any couple (P', p') provided (*a*) that they lie in parallel planes or in the same plane, and (*b*) that their moments are equal, i.e. $P \cdot p = P' \cdot p'$. This allows us to *represent a rotor couple* (P, p) *by a vector perpendicular to the plane of the couple and equal in magnitude to its moment* Pp.

The *sense* of the vector is determined as follows. In the case of infinitesimal rotations it appears from Arts. 67 and 64 that a couple of the type A, Fig. 24, produces a translation upwards from the plane of the figure, *i.e.* towards the reader; while a couple of the type B produces a downward translation, away from the reader. Regarding the couples as rigid figures, their rotors as forces, and the middle point of their arms as fixed, the type A tends to produce rotation in the counter-clockwise, positive, sense; the type B in the negative sense. The former is therefore regarded as positive, and its vector is drawn from its plane towards the reader.

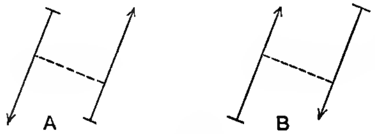

Fig. 24.

72. Let us now return to our infinitesimal displacements.

An infinitesimal translation ds *can be combined with an infinitesimal rotation* dθ *about an axis* l *at right angles to* ds (Fig. 25). To find the resultant single rotation we have only to replace the translation ds by an equivalent couple whose angle of rotation we select equal to that of the given rotation; that is, we put $ds = L_1L_2 \cdot d\theta$, whence

$$L_1L_2 = \frac{ds}{d\theta}.$$

The plane of the couple, being perpendicular to ds, can be taken so as to contain the axis l of the given rotation $d\theta$; and in this plane the couple can be so placed that one of its sides (see Fig. 25) falls into this axis l. Selecting the proper side of the couple, we shall have on l two equal and opposite rotations $d\theta, -d\theta$, which destroy each other, leaving only the rotation $d\theta$, about an axis at the distance $L_1L_2 = ds/d\theta$ from l.

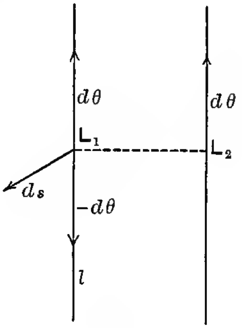

Fig. 25.

Thus it is seen that *the combination of an infinitely small rotation* $d\theta$, *with an infinitely small translation* ds *at right angles to the axis of rotation, produces a single rotation of the same angle about a parallel axis at a distance* $ds/d\theta$ *from the original axis in the plane through this axis perpendicular to the direction of translation.*

73. Exercises.

(1) The telescope of a theodolite, originally horizontal and pointing north, is tipped into an elevation of 60°, and then turned into the prime vertical so as to point west. What single rotation is equivalent to the two successive rotations?

(2) In the preceding example, what would be the result of inverting the order of the two rotations?

(3) The motion of a man in walking may be approximately described as consisting at every step of two rotations of the body about parallel axes perpendicular to the direction of motion, one axis passing through the hip-joint, the other through the foot that remains on the ground while the other foot is thrown forward. Find the angle of swing (assuming the two rotations to be equal and opposite) if the length of the step is 15 inches and the height of the hip-joint $3\frac{1}{4}$ feet.

3. SCREW MOTIONS; TWISTS.

74. We have seen in Arts. 40, 41 that a twist, *i.e.* a rotation combined with a translation parallel to the axis of rotation, constitutes the most general form of the displacement of a rigid body. We proceed to discuss the most important cases of the compositions of rotations and translations resulting in twists.

75. A rotation of angle θ about an axis l can be combined with a translation whose vector is s, by resolving s into two components; s_1 perpendicular to l, and s_2 parallel to l. The former component combines (by Art. 65) with the rotation into a single rotation of the same angle θ about an axis parallel to l. The result is therefore a rotation accompanied by a translation s_2 parallel to the axis of rotation, *i.e.* a twist.

76. If the rotation $d\theta$ and the translation ds are infinitesimal, the axis of the resulting twist has (by Art. 68) a distance $ds/d\theta$ from the axis l of the rotation $d\theta$ and lies in the plane laid through l at right angles to ds.

77. Skew Axes. *The resultant of two successive rotations* θ_1 *and* θ_2 *about two skew axes* l_1 *and* l_2 *is a twist.* This follows of course from the proposition of Art. 40. The axis of the resulting twist is the central axis of the displacement; its direction and position can be found as in Art. 42. Fig. 26 illustrates the process. L_1L_2 is the shortest distance of the axes l_1, l_2. The first rotation, θ_1, about l_1, brings l_2 into its final position l'_2, and L_2 into L'_2; the second rotation, θ_2 about l'_2, brings l_1 into its final position l'_1, and L into L'_1. The axis l of the resulting twist will evidently be the shortest distance of the bisectors of the angles $L_2L_1L'_2$ and $L_1L'_2L'_1$. For a rotation about this line l brings l_2 into l'_2 and l_1 into l'_1.

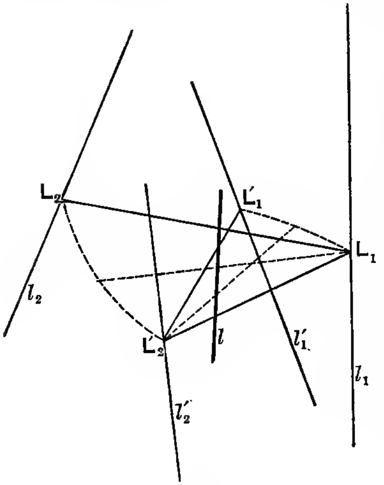

Fig. 26.

78. The angle of the resulting twist is the same as the angle of the rotation resulting from two rotations θ_1, θ_2 about two *intersecting* axes parallel to the given axes l_1, l_2. For (by Art. 65) either one of the rotations, say θ_2 about l_2, may be replaced by a rotation of the same angle θ_2 about an axis parallel to l_2 and intersecting l_1, combined with a translation at right angles to l_2. The two rotations about the intersecting axes can then be combined into a single rotation, and the angle and direction of the axis of this latter rotation are not changed by combination with the translation (Art. 74).

79. It follows from the two preceding articles that a twist can always be resolved into two rotations about skew axes, and this can be done in an infinite number of ways. It is also easy to see that two, or any number of, successive twists can be combined into a single twist by resolving each twist into its rotation and translation, and combining all rotations into a resulting twist and all translations into a resulting translation; the resulting twist combined with the resulting translation gives the twist equivalent to all the given twists.

80. For a more complete account of the geometry of motion the student is referred to A. SCHOENFLIES, *Geometrie der Bewegung*, Leipzig, Teubner, 1886; and to W. SCHELL, *Theorie der Bewegung und der Kräfte*, Leipzig, Teubner, Vol. I., 1879, pp. 144–187. See also R. S. BALL, *Theory of screws*, Dublin, Hodges, 1876; and H. GRAVELIUS, *Ball's theoretische Mechanik starrer Systeme*, Berlin, Reimer, 1889,—for the more advanced parts of the subject. Many authors treat the geometry of motion in connection with Kinematics; see the references in Chapter II., in particular the works of Burmester, Resal, Villié.

Applications to mechanism and machinery will be found in F. REULEAUX, *Kinematics of machinery, edited by A. B. W. Kennedy*, London, Macmillan, 1876; in J. H. COTTERILL, *Applied mechanics*, London, Macmillan, 1884, pp. 99–134; and in ALEX. B. W. KENNEDY, *The mechanics of machinery*, London, Macmillan, 1886.

CHAPTER II.

KINEMATICS.

I. *Time.*

81. Before introducing the idea of time into the study of motion, a word must be said on the measurement of time.

It is the province of astronomy to devise methods for measuring time; the usual method consists in transit observations. Thus the fundamental unit of time in astronomy, or the **sidereal day,** is the interval between two successive upper transits of the true vernal equinox over the same meridian.

82. For the purposes of every-day life, it is more convenient to make the measurement of time depend on the apparent revolution of the sun. But the interval between two successive upper transits of the sun over the same meridian, which is the *true,* or *apparent solar day,* is not constant throughout the year, owing to the inclination of the earth's axis to the plane of its orbit and to the ellipticity of this orbit. The true solar day is thus not well adapted to serve as a unit of time.

Astronomers imagine, therefore, a so-called *first mean sun* moving uniformly in the ecliptic so as to pass the perigee simultaneously with the real sun; and a *second mean sun* moving uniformly in the equator so as to pass the vernal equinox simultaneously with the first mean sun. The interval between two successive passages of the second mean sun over the same meridian is called the **mean solar day.** This may be regarded as the standard on which all time-determinations in mechanics are based.

The mean solar day is subdivided into 24 hours = 1440 minutes = 86 400 seconds. In theoretical mechanics the second is generally used as the unit of time.

83. To reduce mean time to apparent time, it is only necessary to subtract from mean time the so-called *equation of time*, whose value for any particular date is given in the Ephemeris.

84. The relation between mean solar time and sidereal time is readily found by considering that the tropical year, *i.e.* the interval between two successive passages of the sun through the mean vernal equinox, has 365.2422 mean solar days, and of course just one more sidereal day. Hence 1 solar day $= 366.2422/365.2422 = 1.002738$ sidereal day; in other words, the sidereal day contains 86 164.1 seconds of mean time, while the solar day contains 86 400 such seconds.*

85. It will have been noticed that all these methods of measuring time are ultimately based on the assumption that the rotation of the earth on its axis is perfectly uniform. Observation shows this assumption to be true, or at least to have a very high degree of approximation.

It might be asked how we can know, without using some unit of time for comparison, that the earth's rotation on its axis is uniform; in other words, that the mean solar day is constant. Our absolute unit of time would seem to be obtained by reasoning in a circle. This objection is not quite without foundation; and as similar difficulties arise in the case of other fundamental data of mechanics, it may be well to consider the matter a little more in detail.

86. The simplest answer is that we *assume* the constancy of the mean solar day and find this assumption fully justified by the fact that while the whole structure of the astronomical and physical sciences rests on this assumption, the theoretical predictions of these sciences are found to be in close agreement with the results of direct observation.

Historically, the assumption was originally adopted on account of its simplicity, as a practical working hypothesis, and it was found to work well. From the logical point of view we may strengthen its probability by the following considerations.

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* For further particulars see W. CHAUVENET, *Spherical and practical astronomy*, Vol. I., p. 52 sq. and pp. 651–654; also the *American Ephemeris and Nautical Almanac*.

87. The origin of our notion of time as a measurable quantity lies in the subjective sensation that teaches us instinctively to distinguish between shorter and longer intervals of time. This feeling of time is of course (just as in the analogous case of muscular force) far too vague and indefinite to admit of measurement. But it is sufficient to convince us that, approximately, the lengths of successive days are equal. With far greater approximation can we judge by our time-feeling that the oscillations of the pendulum of a clock are nearly isochronous. Let us combine these two entirely independent facts. Careful observation will show that the number of oscillations made by the pendulum in the interval between two culminations of the mean sun is almost precisely the same for every mean day. Moreover, the agreement becomes the more perfect the more we eliminate any causes that tend to disturb the isochronism of the pendulum. It will therefore be reasonable to conclude that the mean solar day must have a very nearly constant length.

But it is to be kept in mind that this is an empirical fact and hence not absolutely true, but only within the limits of the errors of observation. Indeed, certain considerations concerning the friction caused by the tides make it probable that the angular velocity of the earth is diminishing very slowly.*

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* See O. RAUSENBERGER, *Analytische Mechanik*, I., Leipzig, Teubner, 1888, p. 14; H. STREINTZ, *Physikalische Grundlagen der Mechanik*, Leipzig, Teubner, 1883, p. 81 sq.; E. BUDDE, *Allgemeine Mechanik*, I., Berlin, Reimer, 1890, p. 33; THOMSON and TAIT, *Natural philosophy*, I., London, Macmillan, 1879, p. 460; J. C. MAXWELL, *Matter and motion*, New York, Van Nostrand, 1878, p. 27 and p. 60.; K. PEARSON, *Grammar of science*, London, Scott, 1892, pp. 217–230.

II. *Linear Kinematics.*

I. UNIFORM RECTILINEAR MOTION; VELOCITY.

88. Consider a point moving in a straight line. If throughout the whole motion equal spaces are always described in equal times, the motion is said to be **uniform.**

89. Next consider two points each moving uniformly in a straight line. The motions may still be different; for it is possible that while one of the points moves in a given time t over a space s_1, the other moves during the same time t over a different space s_2. The points are then said to have different velocities, and their velocities are said to be as s_1 is to s_2. The **velocity** v of uniform motion is therefore measured by the ratio of the space s described in any time t to this time; that is, $v=s/t$.

90. This equation written in the form

$$s=vt \tag{1}$$

is called the **equation of motion** of the point. It follows from Art. 89 that in uniform motion the velocity v is constant.

With t as abscissa and s as ordinate (or *vice versa*), the equation of uniform motion (1) represents a straight line; the tangent of the angle made by this line with the axis of t represents the velocity v.

91. Let the point P start at the time $t=0$ from a point O (Fig. 27); let it reach the point P_0 at the time $t=t_0$ and the point P_1 at the time $t=t$. Then, putting $OP_0=s_0$, $OP_1=s$, the space passed over in the time $t-t_0$ is $s-s_0$; hence the velocity $v=(s-s_0)/(t-t_0)$. The equation of uniform motion can therefore be written in the form

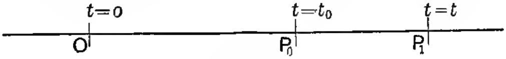

Fig. 27.

$$s-s_0=v(t-t_0). \tag{1'}$$

If the times be counted from the instant when the moving point is at P_0, we have $t_0=0$, and the equation of motion is

$$s=s_0+vt. \qquad (1'')$$

Finally, if both times and spaces are counted from P_0 as origin, we have $s_0=0$, so that $(1'')$ reduces to (1).

92. To measure velocities we must adopt a **unit of velocity.**

In kinematics, the only **fundamental,** *i.e.* independent, units required are those of length and time. All other quantities can be expressed in terms of length and time, and their units are therefore called **derived** units.

Thus, the definition of the velocity of uniform motion as a length divided by a time (Art. 89) can be expressed by the symbolic equation

$$\mathbf{V}=\frac{\mathbf{L}}{\mathbf{T}}, \text{ or } \mathbf{V}=\mathbf{L}\mathbf{T}^{-1},$$

and we say that the **dimensions** of velocity are 1 in length and −1 in time.

When $\mathbf{L}=\mathbf{1}$ and $\mathbf{T}=\mathbf{1}$, we have $\mathbf{V}=\mathbf{1}$. We must therefore select for our unit of velocity that velocity with which unit length is described in unit time.

Hence in the C. G. S. system (see Arts. 13, 14) the unit velocity is a velocity of 1 cm. per second; in the F. P. S. system it is a velocity of 1 ft. per second.

93. In practice other units are often used, and the same concrete velocity can therefore be expressed by different numbers. Thus the same velocity of a railroad train can be described as 30 miles per hour, or 44 ft. per second, or (approximately) 13.41 metres per second.

The symbols s, v, t, etc., in the kinematical equations must be understood to represent the numerical **ratios** of the concrete quantities to their respective units. The symbol v, for instance, stands for the ratio V/V_1 of the concrete velocity V to its unit

V_1, and we have of course the proportion: 30 miles an hour is to 1 mile an hour as 44 ft. per second is to 1 ft. per second, etc.

94. The full meaning of the equation of dimensions $\mathbf{V}=\mathbf{LT}^{-1}$ is obtained if we substitute V/V_1 for $\mathbf{V}$, L/L_1 for $\mathbf{L}$, T/T_1 for $\mathbf{T}$, where V, L, T are the concrete quantities and V_1, L_1, T_1 their units. We find

$$\frac{V}{V_1}=\frac{L}{L_1}\cdot\frac{T_1}{T};$$

and this equation shows two things which are of frequent application in reductions between different systems of units:

(*a*) The *numerical value* V/V_1 of a velocity varies directly as the unit of time and inversely as the unit of length;

(*b*) the *unit* of velocity V_1 varies directly as the unit of length and inversely as the unit of time.*

95. In speaking of velocities, the time unit (usually the second) is frequently understood without being mentioned. This has led to considering velocity as a length (viz. the length passed over in unit time); it can then be represented graphically by a segment of a straight line, and if in addition we combine with the idea of velocity that of the *direction* and *sense* of the motion, its geometrical representative will be a vector (see Art. 45). We shall see later that this view is of particular advantage in studying the velocity of curvilinear motion.

Some recent writers on mechanics use the term *velocity* exclusively in this meaning, *i.e.* as denoting a vector, and apply the term *speed* to denote the numerical magnitude of this vector. In linear kinematics the direction is given, and the "speed" alone is the subject of investigation. The + or − sign of the "speed" expresses the *sense* of the motion.†

96. Exercises.

(1) A train leaves the station A at 9 h. 5 m., passes (without stop-

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* See J. D. Everett, *C. G. S. system of units*, 1891, p. 3.

† See *Syllabus of elementary dynamics*, Part I., prepared by the Association for the Improvement of Geometrical Teaching; London, Macmillan, 1890, p. 8.

ping) B at 9 h. 31 m., C at 9 h. 47 m., and arrives at D at 9 h. 59 m., the distance AD being 36.9 miles. Considering the motion as uniform:

(*a*) What is the velocity?

(*b*) What is the equation of motion?

(*c*) What are the distances BD and CD?

(*d*) If after stopping 5 minutes at D the train goes on with the same velocity as before, when will it reach E, $10\frac{1}{4}$ miles beyond D?

(*e*) Construct a graphical time-table, taking the times as abscissas and the distances as ordinates.

(2) Interpret equations (1′) and (1″) geometrically.

(3) A train leaves Detroit at 9 h. 5 m. A.M. and reaches Chicago at 4 h. 30 m. P.M.; another train leaves Chicago at 12 h. 20 m. and arrives in Detroit at 7 h. 25 m. P.M. The distance is 285 miles. Regarding the motion as uniform and neglecting the stops, find, both analytically and graphically, when and where the trains will meet.

(4) Reduce the following velocities to F. P. S. units: (*a*) Walking 4 miles an hour; (*b*) trotting a mile in 2 m. 10 s.; (*c*) railroad train, from 30 to 50 miles per hour; (*d*) bicyclist, 2 miles in 4 m. $59\frac{3}{5}$ s.; (*e*) sound in air, 333 metres per second.

(5) What is the numerical value of a velocity of 22 ft. per second when the hour is taken as unit of time and the mile as the unit of length?

(6) How is the unit of velocity changed if the minute be adopted as unit of time, the unit of length remaining unchanged?

(7) The mean distance of the sun being $92\frac{1}{3}$ million miles, what is the velocity of light if it takes light 16 m. 40 s. to cross the earth's orbit?

(8) Two trains are running on the same track at the rate of 25 and 15 miles per hour, respectively. If at a certain instant they are 10 miles apart, find when they will collide (*a*) if they are headed the same way; (*b*) if they run in opposite directions.

(9) In what latitude is a bullet shot west with a velocity of 1320 ft. per second at rest relatively to the earth's axis, the radius being taken as 4000 miles?

(10) Two trains, one 250, the other 440 ft. long, pass each other on parallel tracks in opposite directions with equal velocity. A passenger in the shorter train observes that it takes the longer train just 4 seconds to pass him. What is the velocity?

2. VARIABLE RECTILINEAR MOTION; ACCELERATION.

97. If the motion be not uniform, the definition of velocity given in Art. 89 is not applicable, as it would not give any definite meaning to the term.

We may of course divide the whole space, or any portion of it, by the corresponding time; and the quotient so obtained is called the **mean,** or **average, velocity for that space or time.** But its value is in general different for different portions of the path. It simply represents that constant velocity with which the space could be described in the same time in which it is actually described.

98. While we cannot speak, generally, of *the velocity* of a variable motion, we attach a perfectly definite meaning to the expression: *the velocity* of the motion *at any particular point or instant.* We mean by this that constant velocity with which the moving point would continue its motion from that point or instant if thereafter nothing should change its motion.

99. To obtain a mathematical expression for this velocity at the point P, or at the time t, let us consider a point moving in a straight line. Let P (Fig. 28) be its position at the time t, P' its position at the time $t+\Delta t$; let the spaces be counted from the point O as origin so that $OP=s$, $OP'=s+\Delta s$; finally let v and $v+\Delta v$ be the velocities at the times t and $t+\Delta t$. The interval Δt we assume to be so small that Δv does not change sign during the time Δt.

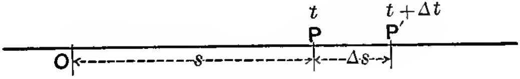

Fig. 28.

Now if during the time Δt the velocity were constant and $=v$, the space described would be $v\Delta t$; if the velocity remained $=v+\Delta v$ during the same interval, the space described would be $(v+\Delta v)\Delta t$. The space Δs actually described during the time Δt with the varying velocity v must evidently lie between these limits; that is, we must have

$$v\Delta t \lesseqgtr \Delta s \lesseqgtr (v+\Delta v)\Delta t,$$

or
$$v \lesseqgtr \frac{\Delta s}{\Delta t} \lesseqgtr v+\Delta v.$$

As Δt, Δs, Δv simultaneously approach the limit 0, we find

$$v=\frac{ds}{dt}; \qquad (2)$$

i.e. in variable motion in a straight line the velocity at any particular point of the path, or at any instant of time, is the value, at that point or time, of the first derivative of the space with respect to time; in other words, *velocity is the time-rate of change of space.*

This definition includes uniform motion as a special case; for in this case, v being constant, the equation of uniform motion (1), Art. 90, gives $ds/dt=v$.

100. Equation (2) enables us to find the velocity if the space be given as a function of the time, say $s=f(t)$; and conversely, if the velocity be given as a function of time or space, we find by integrating the differential equation $ds/dt=v$ an integral *equation of motion* $s=f(t)$. Taking t as abscissa and s as ordinate this equation gives a graphical representation of the motion (compare Art. 90); and it follows from equation (2) that the velocity at any point is represented by the trigonometrical tangent of the angle made by the tangent to the curve at the point with the axis of t.

101. If v be given as function of t, say $v=\phi(t)$, we find from (2) $ds=vdt$, and hence by integration

$$s-s_0=\int_{t_0}^{t} vdt, \qquad (3)$$

where s_0 is the space described during the time t_0. The equation $v=\phi(t)$ furnishes a graphical representation of the velocity, and formula (3) shows that the space $s-s_0$ described during the time $t-t_0$ is represented by the area included between the curve $v=\phi(t)$, the axis Ot, and the ordinates v_0 and v corresponding to t_0 and t, respectively (Fig. 29).

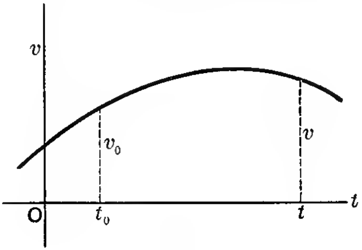

Fig. 29.

102. Similarly, if v be given as a function of s, say $v=\psi(s)$, we have from (2) $dt=ds/v$, and hence

$$t-t_0=\int_{s_0}^{s}\frac{ds}{v}. \qquad (4)$$

The two velocity curves $v=\phi(t)$ and $v=\psi(s)$ are of course in general different, and must not be confounded with the *path* of the moving point, which is here supposed rectilinear.

103. We have seen (Art. 91, equation (1'')) that in the case of uniform motion the velocity $v=(s-s_0)/t$, *i.e.* the rate of change of space with time, is constant. The simplest case of variable motion is that in which the velocity varies uniformly. *The rate at which the velocity varies with the time* is called the **acceleration**; we shall denote it by j.

If the velocity vary uniformly, the acceleration is constant, and we have $j=(v-v_0)/t$, where t is the time during which the velocity changes from v_0 to v.

By reasoning analogous to that employed in Art. 99, we find for the acceleration of *any* rectilinear motion at the time t

$$j=\frac{dv}{dt}=\frac{d^2s}{dt^2}; \qquad (5)$$

that is, *in rectilinear motion the acceleration at any point or instant is the value, at that point or instant, of the second derivative of the space with respect to the time.*

Negative acceleration will thus indicate a decreasing velocity.

104. When the acceleration is constant, the motion is said to be **uniformly accelerated.** In the case of variable acceleration we might again consider its rate of change, which may be called the *acceleration of the second order;* and so on. Compare Art. 156.

105. Conformably to the definition of acceleration, its unit is the "cm. per second per second" in the C. G. S. system, and the "foot per second per second" in the F. P. S. system. As it can rarely be convenient to use two different time units in the unit of acceleration (say, for instance, mile per hour per second), it is customary to mention the time unit but once and to speak of an acceleration of so many feet per second, or cm. per second, it being understood that the other time unit is also the second.

For the *dimensions* of acceleration we have (see Art. 92)

$$\mathbf{J} = \mathbf{VT}^{-1} = \mathbf{LT}^{-2}.$$

Denoting, as in Arts. 93, 94, the concrete value of an acceleration by J, its unit by J_1, and similarly for length and time, we have the equation

$$\frac{J}{J_1} = \frac{L}{L_1} \cdot \frac{T_1^2}{T^2},$$

which shows that (*a*) the numerical value J/J_1 of an acceleration varies directly as the square of the unit of time, and inversely as the unit of length; and (*b*) the unit of acceleration, J_1, varies directly as the unit of length, and inversely as the square of the unit of time.

106. Exercises.

(1) A point moving with constant acceleration gains at the rate of 30 miles an hour in every minute. Express its acceleration in F. P. S. units.

(2) At a place where the acceleration of gravity is $g = 9.810$ metres per second, what is the value of g in feet per second?

(3) A railroad train, 10 minutes after starting, attains a velocity of 45 miles an hour; what was its average acceleration during these 10 minutes?

(4) If the acceleration of gravity, $g = 32$ feet per second, be taken as unit, what is the acceleration of the railroad train in Ex. (3)?

3. APPLICATIONS.

107. Uniformly Accelerated Motion. As in this case the acceleration j is constant (see Art. 103), the equation of motion (5)

$$\frac{d^2s}{dt^2} = j, \text{ or } \frac{dv}{dt} = j,$$

can readily be integrated:

$$v = jt + C.$$

To determine the constant of integration C, we must know the value of the velocity at some particular moment of time. Thus, if $v = v_0$ when $t = 0$, we find $v_0 = C$; hence, substituting this value for C,

$$v - v_0 = jt. \tag{6}$$

This equation, which agrees with the definition of j given in Art. 103, gives the velocity at any time t. Substituting ds/dt for v and integrating again, we find $s = v_0t + \frac{1}{2}jt^2 + C'$, where the constant of integration, C', must again be determined from given "initial conditions." Thus, if we know that $s = s_0$ when $t = 0$, we find $s_0 = C'$; hence

$$s - s_0 = v_0t + \tfrac{1}{2}jt^2. \tag{7}$$

This equation gives the space or distance passed over in terms of the time.

108. Eliminating j between (6) and (7), we obtain the relation

$$s-s_0=\tfrac{1}{2}(v_0+v)t,$$

which shows that in uniformly accelerated motion the space can be found as if it were described uniformly with the mean velocity $\frac{1}{2}(v_0+v)$.

109. To obtain the velocity in terms of the space, we have only to eliminate t between (6) and (7); we find

$$\tfrac{1}{2}(v^2-v_0^2)=j(s-s_0). \tag{8}$$

This relation can also be derived by eliminating dt between the differential equations $v=ds/dt$, $dv/dt=j$, which gives $vdv=jds$, and integrating. The same equation (8) is also obtained directly from the fundamental equation of motion $d^2s/dt^2=j$ by a process very frequently used in mechanics, viz. by multiplying both members of the equation by ds/dt. This makes the left-hand member the exact derivative of $\frac{1}{2}(ds/dt)^2$ or $\frac{1}{2}v^2$, and the integration can therefore be performed.

110. The three equations (6), (7), (8) contain the complete solution of the problem of uniformly accelerated motion. For uniformly retarded motion, taking the direction of motion as positive, we have only to write $-j$ for $+j$.

If the spaces be counted from the position of the moving point at the time $t=0$, we have $s_0=0$, and the equations become

$$v=v_0+jt, \tag{6'}$$

$$s=v_0t+\tfrac{1}{2}jt^2, \tag{7'}$$

$$\tfrac{1}{2}(v^2-v_0^2)=js. \tag{8'}$$

111. If in addition the initial velocity v_0 be zero, the point starting from rest at the time $t=0$, the equations reduce to the following:

$$v=jt, \tag{6''}$$

$$s=\tfrac{1}{2}jt^2, \tag{7''}$$

$$\tfrac{1}{2}v^2=js. \tag{8''}$$

112. The most important example of uniformly accelerated motion is furnished by a body falling in vacuo near the earth's surface. Assuming that the body does not rotate during its fall, its motion relative to the earth is a mere translation, and it is sufficient to consider the motion of any one point of the body. It is known from observation and experiment that under these circumstances the acceleration of a falling body is constant at any given place and equal to about 980 cm., or 32 ft., per second per second.; the value of this so-called *acceleration of gravity* is usually denoted by g.

In the exercises on falling bodies (Art. 114) we make throughout the following simplifying assumptions: the falling body does not rotate; the resistance of the air is neglected, or the body falls in vacuo; the space fallen through is so small that g may be regarded as constant; the earth is regarded as fixed, *i.e.* we consider only the relative motion of the body with respect to the earth.

113. The velocity v acquired by a falling body after falling from rest through a height h is found from (8″) as

$$v=\sqrt{2gh}.$$

This is usually called the **velocity due to the height** (or **head**) **h**, while

$$h=\frac{v^2}{2g}$$

is called the **height** (or **head**) **due to the velocity v.**

114. Exercises.

(1) A body falls from rest at a place where $g = 32.2$. Find (*a*) the velocity at the end of the third second; (*b*) the space fallen through in 5 seconds; (*c*) the space fallen through in the fifth second.

(2) If a railroad train, at the end of 2 m. 40 s. after leaving the station, has acquired a velocity of 30 miles per hour, what was its acceleration (regarded as constant)?

(3) Galilei, who first discovered the laws of falling bodies, expressed them in the following form: (*a*) The velocities acquired at the end of the successive seconds increase as the natural numbers; (*b*) the spaces described during the successive seconds increase as the odd numbers; (*c*) the spaces described from the beginning of the motion to the end of the successive seconds increase as the squares of the natural numbers. Prove these statements.

(4) A stone dropped into the vertical shaft of a mine is heard to strike the bottom after t seconds; find the depth of the shaft, if the velocity of sound be given $= c$. Assume $t = 4$ s., $c = 332$ metres, $g = 980$.

(5) A railroad train approaches a station with uniformly retarded motion. During the first two minutes of its retarded motion it makes 3960 ft.; during the next minute 990 ft. (*a*) When will it come to rest? (*b*) What is the retardation? (*c*) What was the initial velocity? (*d*) When will its velocity be 4 miles an hour?

(6) Interpret equations (6) and (7) geometrically.

(7) A body being projected vertically upwards with an initial velocity v_0, (*a*) how long and (*b*) to what height will it rise? (*c*) When and (*d*) with what velocity does it reach the starting-point?

(8) A bullet is shot vertically upwards with an initial velocity of 1600 ft. per second. (*a*) How high will it ascend? (*b*) What is its velocity at the height of 32,000 ft.? (*c*) When will it reach the ground again? (*d*) With what velocity? (*e*) At what time is it 32,000 ft. above the ground? (*f*) Explain the meaning of the double signs wherever they occur in the answers.

(9) With what velocity must a ball be thrown vertically upwards to reach a height of 100 ft.?

(10) A body is dropped from a point A at a height $AB = h$ above the ground; at the same time another body is thrown vertically

upward from the point B, with an initial velocity v_0. (*a*) When and (*b*) where will they collide? (*c*) If they are to meet at the height $\frac{1}{2}h$, what must be the initial velocity?

115. The general problem of rectilinear motion requires the integration of the differential equation

$$\frac{d^2s}{dt^2}=j, \tag{5}$$

where j is a function of s, t, and v, in connection with the equation

$$\frac{ds}{dt}=v. \tag{2}$$

As these two equations involve four quantities t, s, v, j, a third relation between them, say

$$f(t, s, v, j)=0, \tag{9}$$

is always necessary in order to express three of these four quantities in terms of the fourth. Next to the case of uniformly accelerated motion where the relation (9) is simply $j=$const., the most important cases are those when j is given as a function of s, or of v, or of both s and v.

116. Whenever in nature we observe a motion not to remain uniform, we try to account for the change in the character of the motion by imagining a special cause for such change. In rectilinear motion, the only change that can occur in the motion is a change in the velocity, *i.e.* an acceleration (or retardation). The cause producing acceleration or retardation we call **force** (attraction, repulsion, pressure, tension, friction, resistance of a medium, elasticity, cohesion, etc.), and assume it to be proportional to the acceleration. A fuller discussion of the nature of force and its relation to mass will be found in Chapter III., § II. The present remark is only intended to make more intelligible the physical meaning and applications of the problems to be discussed in the following articles.

117. Acceleration inversely proportional to the square of the distance, *i.e.* $j=\mu/s^2$ where μ is a constant (viz. the acceleration at the distance $s=1$) and s is the distance of the moving point from a fixed point in the line of motion.

The differential equation (5) becomes in this case

$$\frac{d^2s}{dt^2}=\frac{\mu}{s^2}; \tag{10}$$

the first integration is readily performed by multiplying both members by ds/dt so as to make the left-hand member the complete derivative of $\frac{1}{2}(ds/dt)^2$ or $\frac{1}{2}v^2$. Thus we find

$$\tfrac{1}{2}v^2=\mu\int\frac{ds}{s^2}+C=-\frac{\mu}{s}+C, \tag{11}$$

where the constant of integration, C, must be determined from the so-called initial conditions of the problem. For instance, if $v=v_0$ when $s=s_0$, we have $\frac{1}{2}v_0^2=-\mu/s_0+C$; hence, eliminating C between this relation and (11),

$$\tfrac{1}{2}(v^2-v_0^2)=-\mu\left(\frac{1}{s}-\frac{1}{s_0}\right). \tag{12}$$

To perform the second integration, we solve this equation for v and substitute ds/dt for v:

$$\frac{ds}{dt}=\sqrt{\frac{1}{s}[(v_0^2+2\mu/s_0)s-2\mu]},$$

or putting $v_0^2+2\mu/s_0=2\mu/\mu'$,

$$\frac{ds}{dt}=\sqrt{\frac{2\mu}{\mu'}}\cdot\sqrt{\frac{s-\mu'}{s}}. \tag{13}$$

Here the variables s and t can be separated, and we find

$$t=\sqrt{\frac{\mu'}{2\mu}}\int\sqrt{\frac{s}{s-\mu'}}\,ds+C'. \tag{14}$$

To integrate, put $s=x^2$. The result will be different according to the signs of μ, μ', and v, which must be determined from the nature of the particular problem.

118. It is an empirical fact that the acceleration of bodies falling in vacuo on the earth's surface is constant only for distances from the surface that are very small in comparison with the radius of the earth. For larger distances the acceleration is found inversely proportional to the square of the distance from the earth's centre.

By a bold generalization Newton assumed this law to hold generally between any two particles of matter; and this assumption has been verified by all subsequent observations. It can therefore be regarded as a general law of nature that any particle of matter produces in every other such particle, each particle being regarded as concentrated at a point, an acceleration inversely proportional to the square of the distance between these points. This is known as *Newton's law of universal gravitation*, the acceleration being regarded as caused by a force of attraction (or repulsion) inherent in each particle of matter.

It is shown in the theory of attraction that the attraction of a spherical mass, such as the earth, on any particle *outside* the sphere is the same as if the mass of the sphere were concentrated at its centre. The acceleration produced by the earth on any particle outside it is therefore inversely proportional to the square of the distance of the particle from the centre of the earth.

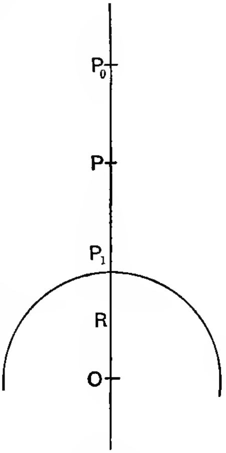

Fig. 30.

119. Let us now apply the general equations of Art. 117 to the particular case of a body falling from a great height towards the centre of the earth, the resistance of the air being neglected.

Let O be the centre of the earth (Fig. 30), P_1 a point on its surface, P_0 the initial position of the moving point at the time $t=0$, P its position at the time t; let $OP_1=R$, $OP_0=s_0$, $OP=s$; and let g be the acceleration at P_1, j the acceleration at P. Then, according to Newton's law, $j:g=R^2:s^2$. This relation determines the value of μ in (10), which becomes

$$\frac{d^2s}{dt^2}=-\frac{gR^2}{s^2}, \tag{15}$$

the minus sign indicating that the acceleration tends to diminish the distances counted from O as origin.

The integration can now be performed as in Art. 117. Multiplying by ds/dt and integrating, we find $\frac{1}{2}v^2=gR^2/s+C$. If the initial velocity be zero, we have $v=0$ for $s=s_0$; hence $C=-gR^2/s_0$, and

$$v=-R\sqrt{2g}\sqrt{\frac{1}{s}-\frac{1}{s_0}}=-R\sqrt{\frac{2g}{s_0}}\sqrt{\frac{s_0-s}{s}}. \tag{16}$$

Here again the minus sign is selected after extracting the square root, since the velocity v is directed in the sense opposite to that of the distance s.

Substituting ds/dt for v, separating the variables v and s, and integrating, we find

$$t=-\frac{1}{R}\sqrt{\frac{s_0}{2g}}\int_{s_0}^{s}\sqrt{\frac{s}{s_0-s}}\,ds$$

$$=\frac{1}{R}\sqrt{\frac{s_0}{2g}}\left\{\sqrt{s(s_0-s)}+s_0\cos^{-1}\sqrt{\frac{s}{s_0}}\right\}. \tag{17}$$

120. Exercises.

(1) Find the velocity with which the body arrives at the surface of the earth if it be dropped from a height equal to the earth's radius, and determine the time of falling through this height.

(2) Interpret equation (17) geometrically.

(3) Show that formula (16) reduces to $v=\sqrt{2gh}$ (Art. 113) when $s=R$ and $s_0-s=h$ is small in comparison with R.

(4) A particle is projected vertically upwards from the earth's surface with an initial velocity v_0. How far will it rise?

(5) If, in (4), the initial velocity be $v_0 = \sqrt{gR}$, how high and how long will the particle rise? How long will it take the particle to rise and fall back to the earth's surface?

(6) A body is projected vertically upwards. Find the least initial velocity that would prevent it from returning to the earth, taking $g = 32$, $R = 4000$ miles.

121. Acceleration directly proportional to the distance, *i.e.* $j = \kappa s$, where κ is a constant and s is the distance of the moving point from a fixed point in the line of motion.

The equation of motion

$$\frac{d^2s}{dt^2} = \kappa s \tag{18}$$

can be integrated by the method used in Art. 117. The result of the second integration will again be different according to the sign of κ. We shall here study only a special case, reserving the general discussion of this law of acceleration for later (see Arts. 177 sq.).

122. It is shown in the theory of attraction that the attraction of a spherical mass such as the earth on any point *within* the mass produces an acceleration directed to the centre of the sphere and proportional to the distance from this centre. Thus, if we imagine a particle moving along a diameter of the earth, say in a straight narrow tube passing through the centre, we should have a case of the motion represented by equation (18).

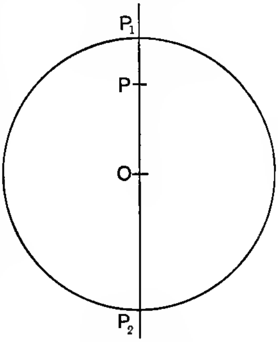

Fig. 31.

To determine the value of κ for our problem we notice that at the earth's surface, that is, at the distance $OP_1 = R$ from the centre O (Fig. 31), the acceleration must be $= g$. If, therefore, j denote the numerical value of the acceleration at any distance $OP = s\,(< R)$, we have $j : g = s : R$, or $j = gs/R$. But the acceleration tends to diminish the distance s, hence $\frac{d^2s}{dt^2} = -\frac{g}{R}s$. Denoting the positive constant g/R by μ^2, the equation of motion is

$$\frac{d^2s}{dt^2} = -\mu^2 s, \text{ where } \mu = \sqrt{\frac{g}{R}}. \tag{19}$$

Integrating as in Arts. 117 and 119, we find

$$\tfrac{1}{2}v^2 = -\tfrac{1}{2}\mu^2 s + C.$$

If the particle starts from rest at the surface, we have $v = 0$ when $s = R$; hence $0 = -\frac{1}{2}\mu^2 R + C$; and subtracting this from the preceding equation, we find

$$v = -\mu\sqrt{R^2 - s^2}, \tag{20}$$

where the minus sign of the square root is selected because s and v have opposite sense.

Writing ds/dt for v and separating the variables, we have

$$dt = -\frac{1}{\mu}\frac{ds}{\sqrt{R^2 - s^2}},$$

whence $$t = \frac{1}{\mu}\cos^{-1}\frac{s}{R} + C'.$$

As $s = R$ when $t = 0$, we have $0 = \frac{1}{\mu}\cos^{-1} 1 + C'$, or $C' = 0$. Solving for s, we find

$$s = R\cos\mu t. \tag{21}$$

Differentiating, we obtain v in terms of t:

$$v = -\mu R \sin\mu t. \tag{22}$$

123. The motion represented by equations (21) and (22) belongs to the important class of *simple harmonic motions* (see Arts. 177 sq.). The particle reaches the centre when $s=0$, *i.e.* when $\mu t=\pi/2$, or at the time $t=\pi/2\mu$. At this time the velocity has its maximum value. After passing through the centre the point moves on to the other end, P_2, of the diameter, reaches this point when $s=-R$, *i.e.* when $\mu t=\pi$, or at the time $t=\pi/\mu$. As the velocity then vanishes, the moving point begins the same motion in the opposite sense.

The time of performing one complete oscillation (back and forth) is called the **period** of the simple harmonic motion; it is evidently

$$T=4\cdot\frac{\pi}{2\mu}=\frac{2\pi}{\mu}.$$

124. Exercises.

(1) Equation (19) is a differential equation whose general integral is known to be of the form

$$s=C_1\sin\mu t+C_2\cos\mu t;$$

determine the constants C_1, C_2 and deduce equations (21) and (22).

(2) Find the velocity at the centre and the period, taking $g=32$ and $R=4000$ miles.

(3) If the acceleration, instead of being directed toward the centre, is directed away from it, the equation of motion would be $d^2s/dt^2=\mu^2 s$. Investigate this motion, which can be imagined as produced by a force of repulsion emanating from the centre.

125. Retardation Due to a Resisting Medium. We know from observation that the velocity of a body moving in a liquid or gas is continually diminished. The resistance of such a medium may be regarded as a force producing a retardation, or negative acceleration. The same may be said of the effect of friction. The law according to which such resistances retard the motion must of course be determined by experiment.

Careful experiments on the resistance offered by the air to the motion of projectiles have shown that this resistance increases with the quantity of air displaced; that is, with the density of the air, the cross-section of the projectile, and the velocity. The retardation due to the resistance of the air can therefore be expressed in the form

$$j = \kappa \rho f(v),$$

where ρ is the density of the air, while κ is a coefficient depending upon the shape, mass, and physical condition of the surface of the projectile. Its value may be regarded as inversely proportional to the mass and directly proportional to the cross-section of the body at right angles to the direction of motion.

The velocity function $f(v)$ may be taken $= cv^2$ for velocities not exceeding 250 metres per second; for greater velocities, up to about 420 metres per second, it is proportional to a higher power of v, or must be represented by a more complicated expression, such as $av^3 + bv + c$; for velocities above 420 metres it seems to be again of the form $c'v^2$.*

126. Assuming the resistance of the air to be proportional to the square of the velocity, the motion of a body falling through air of uniform density is determined by the equation

$$\frac{d^2s}{dt^2} = g - \kappa v^2.$$

To simplify the resulting formulæ, it will be convenient to put $\kappa = \frac{\mu^2}{g}$, so that the equation of motion is

$$\frac{d^2s}{dt^2} = \frac{g^2 - \mu^2 v^2}{g}. \tag{23}$$

Writing $\frac{dv}{dt}$ for $\frac{d^2s}{dt^2}$, the variables v and t can be separated:

$$\frac{g\,dv}{g^2 - \mu^2 v^2} = dt;$$

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* For further particulars the reader is referred to special works on ballistics.

integrating, we find

$$t=\frac{1}{2\mu}\log\frac{g+\mu v}{g-\mu v}, \tag{24}$$

the constant of integration being 0 if the initial velocity be 0. Solving for v, we have

$$v=\frac{g}{\mu}\frac{e^{\mu t}-e^{-\mu t}}{e^{\mu t}+e^{-\mu t}}. \tag{25}$$

As the numerator, apart from a constant factor, is the derivative of the denominator, the second integration can at once be performed, giving

$$s=\frac{g}{\mu^2}\log(e^{\mu t}+e^{-\mu t})+C.$$

For $t=0$, we have $s=0$; hence $0=\frac{g}{\mu^2}\log 2+C$. Hence

$$s=\frac{g}{\mu^2}\log\tfrac{1}{2}(e^{\mu t}+e^{-\mu t}). \tag{26}$$

To find s in terms of v, we may eliminate dt between the differential equations $ds=vdt$ and $dv=\frac{1}{g}(g^2-\mu^2v^2)\,dt$. The resulting equation

$$ds=g\frac{v}{g^2-\mu^2v^2}\,dv$$

is readily integrated; as $v=0$ when $s=0$, we find:

$$s=\frac{g}{2\mu^2}\log\frac{g^2}{g^2-\mu^2v^2}. \tag{27}$$

127. Exercises.

(1) Show that as t increases, the motion considered in Art. 126 approaches more and more a state of uniform motion without ever reaching it.

(2) Show that when μ, and hence κ, becomes 0, the equations of Art. 126 reduce to those for bodies falling in vacuo.

(3) Investigate the motion of a particle thrown vertically upwards in the air with a given initial velocity, the resistance of the air being proportional to the square of the velocity.

(4) Find the whole time of ascent in (3) and the height to which the particle rises.

(5) Show that owing to the resistance of the air a particle thrown vertically upwards returns to the starting point with a velocity less than the initial velocity of projection.

(6) A particle begins moving with an initial velocity v_0 in a medium of constant density whose resistance is proportional to the velocity. Express s and v in terms of t, and v in terms of s.

(7) A body falls from rest in a medium whose resistance is proportional to the velocity. Investigate its motion.

4. ROTATION; ANGULAR VELOCITY; ANGULAR ACCELERATION.

128. A motion of rotation about a fixed axis can be treated in precisely the same way in which we have treated rectilinear motion in the preceding sections. It is only to be remembered that rotations are measured by angles (see Arts. 11–15), while translations are measured by lengths.

129. The rotation of a rigid body (see Art. 8) about a fixed axis is said to be *uniform* if the circular arcs described by the same point in equal times are equal throughout the whole motion; in other words, if the angle of rotation is proportional to the time in which it is described. In this case of uniform rotation, the quotient obtained by dividing the angle of rotation, θ, by the corresponding time, t, is called the **angular velocity**. Denoting it by ω we have $\omega = \theta/t$; and the equation of motion is

$$\theta = \omega t. \tag{1}$$

Thus, expressing the time in seconds and the angle in radians (Art. 15), the angular velocity is equal to the number of radians described per second. (Compare Arts. 88–90.)

130. If the time of a whole revolution be denoted by T, we have, from (1), $2\pi = \omega T$; hence

$$\omega = \frac{2\pi}{T}. \tag{2}$$

In engineering practice it is customary to take a whole revolution as angular unit and to express the angular velocity of uniform motion by the number of revolutions made in the unit of time. Let n, N be the numbers of revolutions per second and per minute, respectively; then we have evidently

$$n = \frac{\omega}{2\pi}, \quad N = \frac{30\,\omega}{\pi}. \tag{3}$$

131. When the rotation is *not* uniform, the quotient obtained by dividing the angle of rotation by the time in which it is described, gives the *mean*, or *average*, *angular velocity* for that time.

The rate of change of the angle of rotation with the time at any particular moment is called the *angular velocity at that moment.* By reasoning in a similar way, as in Art. 99, it will be seen that its mathematical expression is

$$\omega = \frac{d\theta}{dt}. \tag{4}$$

132. The rate at which the angular velocity changes with the time is called the **angular acceleration**; denoting it by α, we have

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}. \tag{5}$$

133. The most important special case of variable angular velocity is that of uniformly accelerated (or retarded) rotation when the angular acceleration is constant. The formulæ for this case have precisely the same form as those given in Arts. 107–111 for uniformly accelerated rectilinear motion. Denoting the constant linear acceleration by j, we have, when the initial velocity is 0,

FOR TRANSLATION:	FOR ROTATION:	
$v = jt,$	$\omega = \alpha t,$	
$s = \frac{1}{2} jt^2,$	$\theta = \frac{1}{2}\alpha t^2,$	(6)
$\frac{1}{2} v^2 = js;$	$\frac{1}{2}\omega^2 = \alpha\theta;$	

and when the initial velocities are v_0 and ω_0, respectively:

FOR TRANSLATION:	FOR ROTATION:	
$v=v_0+jt,$	$\omega=\omega_0+\alpha t,$	
$s=v_0t+\frac{1}{2}jt^2,$	$\theta=\omega_0t+\frac{1}{2}\alpha t^2,$	(7)
$\frac{1}{2}v^2-\frac{1}{2}v_0^2=js;$	$\frac{1}{2}\omega^2-\frac{1}{2}\omega_0^2=\alpha\theta.$	

134. Let a point P, whose perpendicular distance from the axis of rotation is $OP=r$, rotate about the axis with the angular velocity $\omega=d\theta/dt$. In the element of time, dt, it will describe an element of arc $ds=rd\theta=r\omega dt$. Its velocity $v=ds/dt$ (frequently called its **linear** velocity in contradistinction to the angular velocity) is therefore related to the angular velocity of rotation by the equation

$$v=\omega r. \tag{8}$$

135. The radius vector $OP=r$ sweeps over a circular sector which in uniform rotation has an area $S=\frac{1}{2}\theta r^2=\frac{1}{2}\omega tr^2$, while in variable rotation the infinitesimal sector described during the element of time dt is $dS=\frac{1}{2}r^2d\theta=\frac{1}{2}\omega r^2dt$.

The quotients

$$\frac{S}{t}=\frac{1}{2}r^2\frac{\theta}{t}=\frac{1}{2}\omega r^2, \tag{9}$$

for uniform rotation, and

$$\frac{dS}{dt}=\frac{1}{2}r^2\frac{d\theta}{dt}=\frac{1}{2}\omega r^2, \tag{10}$$

for variable rotation, represent, therefore, the *sectorial*, or *areal, velocity*, *i.e.* the rate of increase of area with the time.

The rate of change of this velocity with the time,

$$\frac{d^2S}{dt^2}=\frac{1}{2}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right), \tag{11}$$

is called the *sectorial*, or *areal*, *acceleration*.

136. Exercises.

(1) If a fly-wheel of 12 ft. diameter makes 30 revolutions per minute, what is its angular velocity, and what is the linear velocity of a point on its rim?

(2) A pulley 5 ft. in diameter is driven by a belt travelling 500 ft. a minute. Neglecting the slipping of the belt, find (*a*) the angular velocity of the pulley in radians, and (*b*) its number of revolutions per minute.

(3) Find the constant acceleration (such as the retardation caused by a Prony brake) that would bring the fly-wheel in Ex. (1) to rest in $\frac{1}{3}$ minute.

(4) How many revolutions does the fly-wheel in Ex. (3) make during its retarded motion before it comes to rest?

(5) A wheel is running at a uniform speed of 32 turns a second when a resistance begins to retard its motion uniformly at the rate of 8 radians per second. (*a*) How many turns will it make before stopping? (*b*) In what time is it brought to rest?

(6) A belt runs over two pulleys turning about parallel axes. Show that the angular velocities of the pulleys are inversely proportional to their diameters. Do the pulleys rotate in the same or opposite sense?

III. *Plane Kinematics.*

I. VELOCITY; COMPOSITION OF VELOCITIES; RELATIVE VELOCITY.

137. The motion of a point in a curved path would not be completely characterized by its velocity and acceleration as defined in the preceding section; the varying direction of the motion, and the rate of change of direction, must be taken into account. It is convenient to incorporate these ideas in the definitions of velocity and acceleration. By this generalization of their original meaning, velocity and acceleration become *vectors*, *i.e.* magnitudes having both length and direction.

138. The generalized idea of velocity as a vector may be derived as follows:

Consider a point P moving in a curve (Fig. 32). Let P be its position at the time t, P' its position at the time $t+\Delta t$, and let $P_0P=s$, $PP'=\Delta s$. The space s described in any time t may be regarded as some function of the time t, say $s=f(t)$.

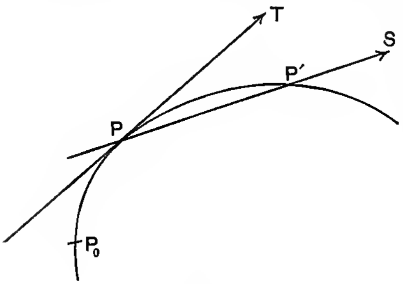

Fig. 32.

The mean velocity $\Delta s/\Delta t$ for the time Δt during which the point passes from P to P' may be represented by a length PS laid off on the chord PP' from P. As Δt diminishes, P' approaches P, and in the limit when $\Delta s/\Delta t$ becomes the derived function $ds/dt=f'(t)$, the chord merges into the tangent at P. This leads us to represent the velocity at the time t, or at the place P, by a length PT proportional to ds/dt laid off on the tangent at P from this point in the sense of the motion. The vector PT will then completely represent the velocity at the time t.

139. The vector PT may also be regarded as the limit of a vector PS laid off on the chord PP' as before, but proportional to the velocity with which the point would describe the *chord* PP' in the time Δt, *i.e.* to the velocity $PS = \frac{\text{chord } PP'}{\Delta t}$. For as Δt approaches the limit 0, PS approaches the direction of the tangent, and the ratio of the arc Δs to the chord PP' approaches the limit 1. Hence the equation $\frac{\Delta s}{\Delta t} = \frac{\Delta s}{\text{chord } PP'} \cdot PS$ gives in the limit $\lim \frac{\Delta s}{\Delta t} = \lim PS$, or $PT = \lim PS$.

It may be noticed here that, in view of the practical applications, the function $f(t) = s$ is in mechanics always supposed to be itself continuous and to possess continuous and finite derivatives of the first and second order.

140. Velocity having thus been defined as a vector, we may at once apply to it the rules for vector composition and vector resolution laid down in Arts. 45–55 for vectors representing displacements. Thus if a point be subjected to two or more simultaneous velocities, the velocity of the resulting motion will be represented by the vector found by geometrically adding the component velocities. A velocity may be resolved into any number of component velocities whose geometrical sum is equal to the given velocity.

141. We proceed to consider the most important cases of resolution of a velocity in a plane.

Let a point P move in a curve P_0P (Fig. 33) whose equation is referred to rectangular Cartesian co-ordinates x, y. It is usually convenient in this case to resolve the velocity v parallel to the axes into v_x and v_y.

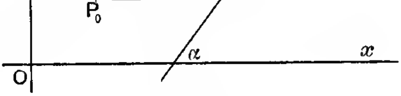

Fig. 33.

If α be the angle made by the vector v with the axis of x, we have $v_x = v \cos\alpha$, $v_y = v \sin\alpha$. And as the element ds of the curve at P makes the same angle α with the axis of x, we also have $dx = ds\cos\alpha$, $dy = ds\sin\alpha$. Dividing by dt and comparing with the preceding equations, we find

$$v_x = \frac{dx}{dt}, \quad v_y = \frac{dy}{dt}. \tag{1}$$

Conversely, knowing the velocities of the moving point parallel to the axes, we find its resulting velocity from the relation

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}. \tag{2}$$

142. If the equation of the path be given in polar co-ordinates, it may be convenient to resolve the velocity v along the radius vector OP and at right angles to it (Fig. 34).

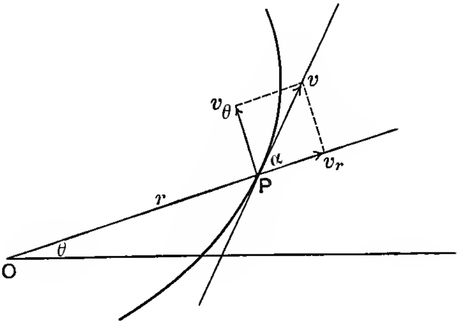

Fig. 34.

Let r, θ be the polar co-ordinates, α the angle between v and r; then $v_r = v\cos\alpha$, $v_\theta = v\sin\alpha$. The element ds of the curve has in the same directions the components $dr = ds\cos\alpha$, $rd\theta = ds\sin\alpha$. Hence, dividing by dt, we find

$$v_r = \frac{dr}{dt}, \quad v_\theta = r\frac{d\theta}{dt}, \tag{3}$$

and

$$v = \sqrt{v_r^2 + v_\theta^2} = \sqrt{\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2}. \tag{4}$$

143. In the case of **relative motion** we have to distinguish between the *absolute velocity* v of a point, its *relative velocity* v_1, and the *velocity of the body of reference* v_2.

To fix the ideas, imagine a man walking on deck of a steamboat. His velocity of walking is his relative velocity v_2; the velocity of the boat (say with respect to the water or shore regarded as fixed), or more exactly speaking, the velocity of that point of the boat at which the man happens to be at the time, is the velocity v_2 of the body of reference; and the velocity with which the man is moving with respect to the water or shore, is his absolute velocity.

Representing these three velocities by means of their vectors, we evidently *find the absolute velocity* v *as the geometric sum of the relative velocity* v_1 *and the velocity* v_2 *of the body of reference,* just as in the case of displacements of translation (Art. 53). And conversely, *the relative velocity is found by geometrically subtracting from the absolute velocity the velocity of the body of reference.*

It is often convenient to state the last proposition in a somewhat different form. Imagine that we give the velocity $-v_2$ both to the man and to the boat; then the boat is brought to rest, and the resulting velocity of the man is what was before his relative velocity. Hence *the relative velocity is found as the resultant of the absolute velocity, and the velocity of the body of reference reversed.*

144. Exercises.

(1) A straight line in a plane turns with constant angular velocity ω about one of its points O, while a point P, starting from O, moves along the line with a constant velocity v_0. Determine the absolute path of P and its absolute velocity v.

(2) Show how to construct the tangent and normal to the spiral of Archimedes $r = a\theta$, where $\theta = \omega t$.

(3) A wheel of radius a rolls on a straight track with constant velocity (of its centre) v_0. Find the velocity v of a point P on the rim.

(4) Show that the tangent to the cycloid described by P, Ex. (3), passes through the highest point of the wheel.

(5) Show that the tangent to the ellipse bisects the angle between the radii vectores r, r' drawn from any point P on the ellipse to the foci S, S'.

(6) Construct the tangent to any conic section when a directrix and the corresponding focus are given.

(7) Two trains of equal length pass each other with equal velocity on parallel tracks. A man riding on a bicycle along the track at the rate of 8 miles an hour notices that the train meeting him takes 4 seconds to pass him, while the other takes 6 minutes. Find the velocity of the trains.

(8) A swimmer, starting from a point A on one bank of a river, wishes to reach a certain point B on the opposite bank. The velocity v_2 of the current and the angle θ made by AB with the direction of the current being given, determine the least relative velocity v_1 of the swimmer in magnitude and direction.

(9) Two men, A and B, walking at the rate of 3 and 4 miles an hour, respectively, cross each other at a rectangular street corner. Find the relative velocity of A with respect to B in magnitude and direction.

(10) A man jumps from a car at an angle of 60° with a velocity of 8 feet a second (relatively to the car). If the car be running 10 miles an hour, with what velocity and in what direction does the man strike the ground?

(11) The point P_1 moves with constant velocity v_1 along the line P_1Q. In what direction P_2Q must a point P_2 move with constant velocity v_2 in order to meet P_1? What is the locus of Q when the direction of P_1Q varies? When is the solution impossible?

(12) A point P moves uniformly in a circle, while another point Q moves with equal velocity along a tangent to the circle. Find the relative path of either point with respect to the other.

(13) The velocity of light being taken as 300,000 kilometres per second, and the velocity of the earth in its orbit as 30 kilometres, determine approximately the constant of the annual aberration of the fixed stars.

2. APPLICATIONS.

145. The motion of the **piston of a steam engine** furnishes interesting illustrations of the application of graphical methods in kinematics.

In Fig. 35, let $OQ=a$ be the crank arm, $PQ=l=ma$ the connecting rod, $P_1P_2=s$ the "stroke," so that $l=ma=\frac{1}{2}ms$.

As $P_1P_2 = A_1A_2 = 2a$, we may regard A_1A_2 as representing the stroke. The position of the piston head P at the time when the crank pin is at Q will then be found as the intersection N of a circle of radius l described about P with the diameter A_1A_2 of the crank circle; in other words, N represents the position of the piston corresponding to the angle $A_1OQ = \theta$ in the forward stroke and to the angle $A_1OQ' = 2\pi - \theta$ in the return stroke.

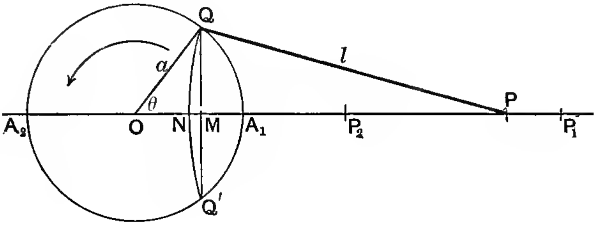

Fig. 35.

146. The crank may generally be assumed to turn uniformly, making n revolutions per second. The linear velocity of the crank pin Q is therefore $u = 2\pi a \cdot n = \pi ns$.

For the piston head P, or for the point N, we must distinguish between its *mean*, or *average*, velocity V, and its variable *instantaneous* velocity v at any particular moment. For each revolution of the crank the piston head completes a double stroke so that its mean speed is $V = 2ns$. Hence we have

$$\frac{u}{V} = \frac{\pi ns}{2ns} = \frac{\pi}{2}.$$

147. The instantaneous velocity v of the piston can be found graphically by considering the motion of the connecting rod PQ. The velocity u of the end Q is known, both in magnitude and direction; the velocity v of the other end is known in direction only. Now considering that the length of the rod PQ is invariable and hence the components of u and v along PQ must be equal, we can find the magnitude of v by drawing (Fig. 36) from any point M parallels to u and v, laying off u to scale and drawing through its extremity a perpendicular to the direction of PQ; this perpendicular will cut off the proper length on the direction of v.

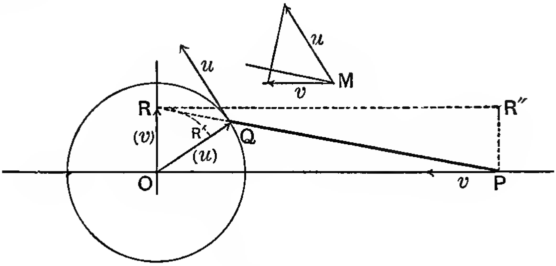

Fig. 36.

In applying this construction to our case it will be convenient to turn the auxiliary diagram of velocities by an angle of 90° and place it so as to make M coincide with O; u will then lie along OQ, and v at right angles to OP. Hence, if the scale of velocities be selected so as to have u represented in length by OQ, v will be represented on the same scale by OR, that is, by the segment cut off by PQ produced on the perpendicular to OP drawn through O.

148. The variation of the piston velocity in the course of the motion can best be exhibited graphically. Thus a *polar curve of piston velocity* is obtained by laying off on OQ a length $OR' = OR$, for a number of positions of OQ, and joining the points R' by a continuous curve.

Another convenient method consists in erecting perpendiculars to OP at the various positions of P and laying off, on these perpendiculars, $OR'' = OR = v$.

149. To derive an analytical expression for the piston velocity v, let ϕ be the angle OPQ which determines the position of the connecting rod.

It follows from the construction of the velocity v given in Art. 147 (see Fig. 36) that

$$\frac{v}{u}=\frac{OR}{OQ}=\frac{\sin(\theta+\phi)}{\cos\phi}=\sin\theta+\cos\theta\tan\phi.$$

If, as is usually the case, the connecting rod is much longer than the crank arm, ϕ will be a small angle, and we may substitute $\sin\phi$ for $\tan\phi$. But from the triangle OPQ we have

$$\frac{\sin\phi}{\sin\theta}=\frac{OQ}{PQ}=\frac{a}{l}=\frac{1}{m}.$$

Hence $v=u\left(\sin\theta+\cos\theta\cdot\frac{1}{m}\sin\theta\right)=u\left(\sin\theta+\frac{1}{2m}\sin 2\theta\right)$.

150. The motion of the piston head being rectilinear, we find its acceleration j by differentiating the expression for v found in the preceding article with respect to t:

$$j=\frac{dv}{dt}=\left(\sin\theta+\frac{1}{2m}\sin 2\theta\right)\frac{du}{dt}+u\left(\cos\theta+\frac{1}{m}\cos 2\theta\right)\frac{d\theta}{dt},$$

or, since $\frac{d\theta}{dt}=\omega=u/a$,

$$j=\left(\sin\theta+\frac{1}{2m}\sin 2\theta\right)\frac{du}{dt}+\left(\cos\theta+\frac{1}{m}\cos 2\theta\right)\frac{u^2}{a},$$

where $\frac{du}{dt}=0$ if the crank motion can be regarded as uniform.

151. If the connecting rod were of infinite length so as to make PQ (in Fig. 35) parallel to A_1A_2, the position of the piston corresponding to the position Q of the crank pin would be represented by the projection M of Q on A_1A_2; that is, NM would be $=0$. This length NM is therefore called the *deviation due to the obliquity* of the connecting rod.

With $NM=0$ the expression for the acceleration (Art. 150) would reduce to $dv/dt=(u^2/j)\cos\theta$, representing a simple harmonic motion (see Art. 179).

152. The slide valve of a steam engine is generally worked by an eccentric whose radius is set on the shaft at such an angle as to shut off the steam when the crank makes a certain angle θ with the direction of motion of the piston. It follows that the fraction of stroke completed before cut-off takes place is affected by the obliquity of the connecting rod. The rates of cut-off are therefore different in the forward and backward strokes. In the forward stroke, the effect of the obliquity is to put the piston in advance of the position it would have if the connecting rod were of infinite length; in the return stroke, *i.e.* when θ is greater than π, the piston lags behind.

153. An analytical expression for the deviation due to obliquity is readily obtained from Fig. 35. We have

$$MN = PN - PM = l(1 - \cos\phi)$$

$$= ms\sin^2\frac{\phi}{2} = \frac{ms}{4}\left(2\sin\frac{\phi}{2}\right)^2,$$

or approximately, since ϕ is small,

$$MN = \frac{ms}{4}\sin^2\phi.$$

Also, as in Art. 149, $\dfrac{\sin\phi}{\sin\theta} = \dfrac{1}{m}$;

hence $$MN = \frac{s}{4m}\sin^2\theta.$$

The greatest value of MN is thus seen to be $s/4m$; for instance, if the connecting rod be four times the length of the crank, the deviation due to obliquity cannot exceed 1/16 of the stroke.

154. Exercises.*

(1) Construct a polar diagram exhibiting the position of the piston for all angles θ by laying off on the crank arm OQ a length $ON' = ON$ and joining the points N' by a continuous curve.

(2) Construct the curves of piston velocity indicated in Art. 152.

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* These problems are taken with slight modification from COTTERILL'S *Applied mechanics*, 1884, p. 112.

(3) Show that for a connecting rod of infinite length the two loops of the curve of Ex. 1 reduce to two equal circles.

(4) The driving wheels of a locomotive are 6 ft. in diameter; find the number of revolutions per minute and the angular velocity, when running at 50 miles per hour. If the stroke be 2 ft., find the speed of the piston.

(5) The pitch of a screw is 24 ft., and the number of revolutions 70 per minute. Find the speed in knots. If the stroke is 4 ft., find the speed of piston in feet per minute.

(6) The stroke of a piston is 4 ft., and the connecting rod is 9 ft. long. Find the position of the crank, when the piston has completed the first quarter of the forward and backward strokes respectively. Also find the position of the piston when the crank is upright.

(7) The valve gear is so arranged in the last question as to cut off the steam when the crank is 45° from the dead-points both in the forward and backward strokes. Find the point at which steam will be cut off in the two strokes. Also when the obliquity of the connecting rod is neglected.

3. ACCELERATION IN CURVILINEAR MOTION.

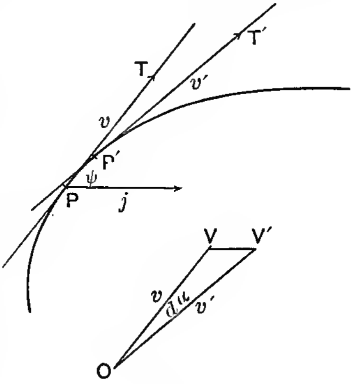

Fig. 37.

155. Let the velocity of a moving point be represented by the vector $v = PT$ at the time t, and by the vector $v' = P'T'$ at the time $t + \Delta t$ (Fig. 37). Then, drawing from any point O OV and OV' respectively equal and parallel to PT and $P'T'$, the vector VV' represents the geometrical difference between v' and v; in other words, VV' is the velocity which, geometrically added to v, produces v'. The vector VV' approaches the limit 0 at the same time with Δt and PP'. This limit of VV' for an infinitely small time dt may be called the *geometrical differential* or *vector differential*, of v.

Dividing this infinitesimal vector by dt, we obtain in general a finite magnitude $\frac{\lim VV'}{dt}$, the *geometrical derivative* of the velocity with respect to the time, and that is what we call the **acceleration** at the time t or at the point P. We represent it geometrically by a vector j drawn from P parallel to the direction of $\lim VV'$.

It will be noticed that the sense of the acceleration will be towards that side of the tangent of the curve on which the centre of curvature is situated.

156. Suppose a point P to move along a curve $P_1P_2P_3 \dots$ with variable velocity v (Fig. 38). From any fixed origin O draw a vector $OV_1 = v_1$, equal and parallel to the velocity v_1 of

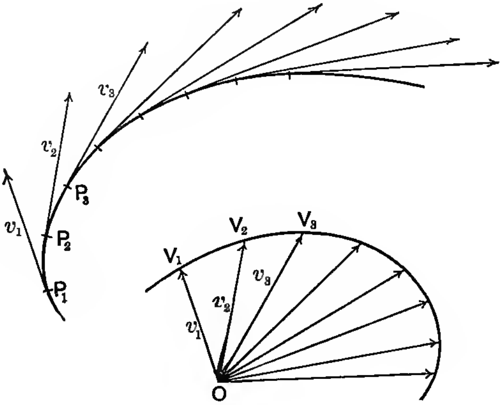

Fig. 38.

P_1, and repeat this construction for every position of the moving point P. The ends V_1, V_2, V_3, ... of all these radii vectores drawn from O will form a continuous curve $V_1V_2V_3 \dots$ which is called the **hodograph** of the motion of P.

If we imagine a point V describing this curve $V_1V_2V_3 \dots$ at the same time that P describes the curve $P_1P_2P_3 \dots$, it is evident that the velocity of V, *i.e.* $\frac{\lim V_1V_2}{dt}$, laid off on the tangent of the curve $V_1V_2V_3 \dots$, represents the acceleration of the point P

both in magnitude and direction; i.e. *the velocity of the hodograph is the acceleration of the original motion.*

It is easy to see how these considerations might be extended. We might construct the hodograph of the hodograph; its velocity might be called the acceleration of the second order for the motion of P; and so on.

It is sometimes convenient to draw the radii vectores of the hodograph not parallel to the velocities of the point P, but so as to make some constant angle (usually a right angle) with these velocities.

157. Exercises.

(1) Discuss rectilinear motion as a special case of plane motion.

(2) Show that the hodograph of rectilinear motion is a straight line.

(3) Show that the velocity of a moving point is increasing, constant, or diminishing, according to the value of the angle ψ between v and j (Fig. 37).

158. Acceleration having been defined as a vector, the rules for vector composition and resolution may be applied to acceleration just as they were before applied to displacements and to velocities. Thus, a point subjected to two or more simultaneous accelerations will have a resulting acceleration found by geometrically adding the component accelerations; and conversely, the acceleration of a point may be resolved in various ways.

159. Let the vector j which represents the acceleration of the point P at the time t, make an angle ψ with the vector representing the velocity v at the same time (see Fig. 37). Resolving the vector j along the tangent and normal at P, we obtain the **tangential acceleration** $j_t = j \cos \psi$ and the **normal acceleration** $j_n = j \sin \psi$.

To find expressions for these components, let us regard PP' in Fig. 37 as the element ds of the path described by P; then the length of $P'T'$, or of OV', is $v' = v + dv$, and the angle VOV', being equal to the angle between two consecutive tangents of the curve, is the angle of contingence $d\alpha$ at P. This angle being equal to the angle between the normals at P and P', we have $\rho d\alpha = ds$, where ρ is the radius of curvature at P.

Resolving the elementary acceleration, *i.e.* the infinitesimal vector VV' along OV and at right angles to OV, we find the components $VV' \cos\psi = dv$, $VV' \sin\psi = v d\alpha = v ds/\rho$. Dividing by dt and observing that $ds/dt = v$, we finally obtain

$$j_t = \frac{dv}{dt}, \tag{1}$$

$$j_n = v\frac{d\alpha}{dt} = \rho\left(\frac{d\alpha}{dt}\right)^2 = \frac{v^2}{\rho}. \tag{2}$$

By composition we have

$$j = \sqrt{j_t^2 + j_n^2} = \sqrt{\left(\frac{dv}{dt}\right)^2 + \frac{v^4}{\rho^2}}. \tag{3}$$

160. When rectangular Cartesian co-ordinates are used, we may resolve the acceleration j into two components $j_x = j\cos\phi$, $j_y = j\sin\phi$ parallel to the co-ordinate axes Ox, Oy; ϕ being the angle made by the vector j with the axis of x. We obtain an expression for j_x by projecting the infinitesimal triangle OVV' (Fig. 37) on the axis Ox and denoting, as before, the projections of the velocities OV, OV' by v_x, v'_x. This gives

$$VV' \cos\phi = v'_x - v_x = dv_x,$$

whence, dividing by dt, $j_x = dv_x/dt$. Similarly, we find $j_y = dv_y/dt$. Hence, by formulæ (1), Art 141,

$$j_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}, \quad j_y = \frac{dv_y}{dt} = \frac{d^2y}{dt^2}. \tag{4}$$

These so-called *equations of motion* offer the advantage that the curvilinear motion is replaced by two rectilinear motions, thus avoiding the use of vectors.

By composition, we have of course

$$j = \sqrt{j_x^2 + j_y^2} = \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2}. \tag{5}$$

161. For polar co-ordinates r, θ, we may resolve the acceleration j into a component j_r along the radius vector r and a component j_θ at right angles to r. Expressions for these components are readily found by projecting the components

$$j_x = \frac{d^2x}{dt^2} \text{ and } j_y = \frac{d^2y}{dt^2}$$

on r and at right angles to r (Fig. 39):

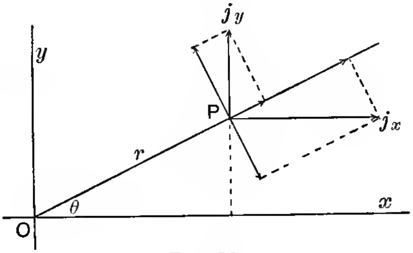

Fig. 39.

$$j_r = \frac{d^2x}{dt^2}\cos\theta + \frac{d^2y}{dt^2}\sin\theta, \quad j_\theta = -\frac{d^2x}{dt^2}\sin\theta + \frac{d^2y}{dt^2}\cos\theta.$$

Differentiating the relations $x = r\cos\theta$, $y = r\sin\theta$, we find

$$\frac{dx}{dt} = \frac{dr}{dt}\cos\theta - r\sin\theta\frac{d\theta}{dt}, \quad \frac{dy}{dt} = \frac{dr}{dt}\sin\theta + r\cos\theta\frac{d\theta}{dt};$$

and differentiating again:

$$\frac{d^2x}{dt^2} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\cos\theta - \left[2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right]\sin\theta,$$

$$\frac{d^2y}{dt^2} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\sin\theta + \left[2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right]\cos\theta.$$

Substituting these expressions for $\frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt^2}$ in the above equations for j_r, j_θ, we find:

$$j_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2, \quad j_\theta = 2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2} = \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right). \tag{6}$$

162. The meaning of these expressions will perhaps be better understood by the following geometrical derivation. As shown in Art. 142, the velocity v has the components

$$v_r = \frac{dr}{dt}, \quad v_\theta = r\frac{d\theta}{dt},$$

the former along the radius vector, the latter at right angles to it. During the element of time dt, while the moving point passes from P to P' (Fig. 40), each of the vectors v_r, v_θ

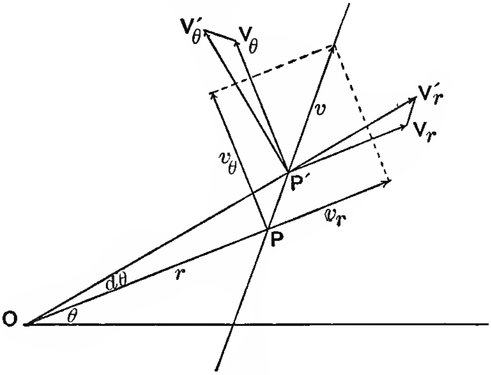

Fig. 40.

receives a geometrical increment $V_r V'_r$, $V_\theta V'_\theta$. Let us resolve each of these infinitesimal vectors along r and at right angles to r, and then combine the two components along r, and also the two components perpendicular to r; finally, dividing by dt, we obtain j_r and j_θ.

Thus v_r gives $\frac{d^2r}{dt^2}$ along r, and $\frac{dr}{dt}\frac{d\theta}{dt}$ at right angles to r, while v_θ, or $r\frac{d\theta}{dt}$, contributes $-r\left(\frac{d\theta}{dt}\right)^2$ along r and

$$\frac{d}{dt}\left(r\frac{d\theta}{dt}\right) = \frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}$$

at right angles to r. We obtain in this way the same expressions for j_r, j_θ as in the formulæ (6) above.

163. Exercises.

(1) Show that the sectorial velocity (Art. 135) is constant whenever $\dot{j}_\theta = 0$.

(2) Show that the normal component of the acceleration is the product of the radius of curvature into the square of the angular velocity about the centre of curvature.

(3) Show that the velocity is the mean proportional between the acceleration and half the chord intercepted by the direction of the acceleration on the osculating circle.

(4) If the acceleration of a point P be always directed to a fixed point A, show that the radius vector AP describes equal areas in equal times.

(5) Show that in uniform circular motion the acceleration is directed to the centre and proportional to the radius.

(6) A wheel rolls on a straight track; find the acceleration of its lowest point.

4. APPLICATIONS.

164. Inclined Plane. Imagine a body sliding down a smooth plane inclined at an angle θ to the horizon. In addition to the assumptions made in the case of falling bodies (see Art. 112) we assume that the motion takes place along a "line of greatest slope," *i.e.* in a vertical plane at right angles to the intersection of the inclined plane with a horizontal plane. A "smooth" plane means one that offers no frictional resistance. The body is therefore subject only to the acceleration of gravity, g; and it is sufficient to consider the motion of a single point of the body.

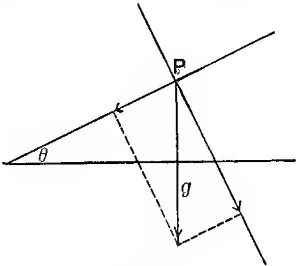

Fig. 41.

Resolving g into two components, $g\cos\theta$ perpendicular to the plane and $g\sin\theta$ along the plane (Fig. 41), it will be seen that the former component, being at right angles to the velocity, cannot change the magnitude of this velocity.

We have therefore simply a rectilinear motion with the constant acceleration $g \sin\theta$, so that all the formulæ of Art. 107–113 will here apply if for the acceleration j (or g) we substitute $g \sin\theta$.

Thus, if the initial velocity be 0, the motion is determined by the equations

$$v = g \sin\theta \cdot t, \qquad (1)$$

$$s = \tfrac{1}{2} g \sin\theta \cdot t^2, \qquad (2)$$

$$\tfrac{1}{2} v^2 = g \sin\theta \cdot s. \qquad (3)$$

165. If there be an initial velocity v_0 parallel to the line of greatest slope of the inclined plane, the equations are

$$v = v_0 + g \sin\theta \cdot t, \qquad (1')$$

$$s = v_0 t + \tfrac{1}{2} g \sin\theta \cdot t^2, \qquad (2')$$

$$\tfrac{1}{2}(v^2 - v_0^2) = g \sin\theta \cdot s, \qquad (3')$$

where v_0 is to be regarded as positive if its direction is down the plane and negative when up the plane.

If the initial velocity v_0 be inclined to the plane at an angle β, it can be resolved into the components $v_0 \cos\beta$ and $v_0 \sin\beta$, the former alone being effective so that $v_0 \cos\beta$ must be substituted for v_0 in the above formulæ.

166. Exercises.

(1) A railroad train is running up a grade of 1 in 250 at the rate of 15 miles an hour when the coupling of the last car breaks. Neglecting friction, (*a*) how far will the car be after two minutes from the point where the break occurred? (*b*) When will it begin moving down the grade? (*c*) How far behind the train will it be at that moment? (*d*) If the grade extend 2000 ft. below the point where the break occurred, with what velocity will it arrive at the foot of the grade?

(2) Show that the final velocity is independent of the inclination of the plane; in other words, in sliding down a smooth inclined plane a

body acquires the same velocity as in falling vertically through the "height" of the plane.

(3) Show that it takes a body twice as long to slide down a plane of 30° inclination as it would take it to fall through the height of the plane.

(4) At what angle θ should the rafters of a roof of given span $2b$ be inclined to make the water run off in the shortest time?

(5) Prove that the times of descending from rest down the chords issuing from the highest (or lowest) point of a vertical circle are equal.

(6) If any number of points starting at the same time from the same point slide down different inclined planes, they will at any time t all be situated on a sphere passing through the starting point and having a diameter $= \frac{1}{2} g t^2$.

(7) Show how to construct geometrically the line of quickest descent from a given point: (a) to a given straight line, (b) to a given circle, situated in the same vertical plane.

(8) Analytically, the line of quickest or slowest descent from a given point to a curve in the same vertical plane is found by taking the equation of the curve in polar co-ordinates, $r = f(\theta)$, with the given point as origin and the axis horizontal. The time of descending the radius vector r is $t = \sqrt{2r/(g \sin\theta)}$. Show that this becomes a maximum or minimum when $\tan\theta = f(\theta)/f'(\theta)$, according as $f(\theta) + f''(\theta)$ is negative or positive.

(9) Show that the line of quickest descent to a parabola from its focus, the axis of the parabola being horizontal, is inclined at an angle of 60° to the horizon.

167. Projectiles. With the same assumptions as in Art. 112, the motion of a projectile reduces to that of a point subject to the constant acceleration of gravity and starting with an initial velocity v_0 inclined to the horizon at an angle ϵ different from 90°. The angle ϵ between the horizon and the initial velocity is called the **elevation** of the projectile.

Taking the horizontal line through the starting point O as axis of x, the vertical upwards as positive axis of y (Fig. 42), the

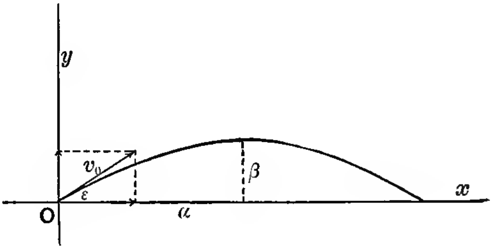

Fig. 42.

x-component of the acceleration is evidently 0, while the y-component is $-g$; hence, by (4), Art. 160, the equations of motion are

$$\frac{d^2x}{dt^2}=0, \quad \frac{d^2y}{dt^2}=-g. \tag{4}$$

The first integration gives

$$\frac{dx}{dt}=C_1, \quad \frac{dy}{dt}=-gt+C_2.$$

As $\dfrac{dx}{dt}=v_x$, $\dfrac{dy}{dt}=v_y$ are the components of the velocity v at the time t, the constants are determined from the values of v_x, v_y at the time $t=0$; viz. $v_0\cos\epsilon=C_1$, $v_0\sin\epsilon=-gt+C_2$. We have therefore

$$v_x\equiv\frac{dx}{dt}=v_0\cos\epsilon, \quad v_y\equiv\frac{dy}{dt}=v_0\sin\epsilon-gt. \tag{5}$$

Integrating again, we find

$$x=v_0\cos\epsilon\cdot t, \qquad y=v_0\sin\epsilon\cdot t-\tfrac{1}{2}gt^2, \tag{6}$$

the constants of integration being 0, since for $t=0$ we have $x=0$, $y=0$.

These values of x, y, v v_y show that the motion in the horizontal direction is uniform while in the vertical direction it is uniformly accelerated. This is otherwise directly evident from the nature of the problem.

Eliminating t between the expressions for x and y, we find the *equation of the path*

$$y = \tan\epsilon \cdot x - \frac{g}{2\,v_0^2\cos^2\epsilon} \cdot x^2, \tag{7}$$

which represents a parabola passing through the origin. To find its vertex and latus rectum, divide by the coefficient of x^2 and rearrange:

$$x^2 - \frac{2\,v_0^2}{g}\sin\epsilon\cos\epsilon \cdot x = -\frac{2\,v_0^2}{g}\cos^2\epsilon \cdot y;$$

completing the square in x, the equation can be written in the form

$$\left(x - \frac{v_0^2}{2g}\sin 2\,\epsilon\right)^2 = -\frac{2\,v_0^2}{g}\cos^2\epsilon\left(y - \frac{v_0^2}{2g}\sin^2\epsilon\right). \tag{7'}$$

The co-ordinates of the vertex are therefore $\alpha = \frac{v_0^2}{2g}\sin 2\,\epsilon$, $\beta = \frac{v_0^2}{2g}\sin^2\epsilon$; the latus rectum $4\,a = \frac{2\,v_0^2}{g}\cos^2\epsilon$; the axis is vertical, and the directrix is a horizontal line at the distance $a = \frac{v_0^2}{2g}\cos^2\epsilon$ above the vertex.

168. Exercises.

(1) Show that the velocity at any time is $v = \sqrt{v_0^2 - 2gy}$.

(2) Prove that the velocity of the projectile is equal in magnitude to the velocity that it would acquire by falling from the directrix: (*a*) at the starting point, (*b*) at any point of the path (see Art. 113).

(3) Show that a body projected vertically upwards with the initial velocity v_0 would just reach the common directrix of all the parabolas described by bodies projected at different elevations ϵ with the same initial velocity v_0.

(4) The *range* of a projectile is the distance from the starting point to the point where it strikes the ground. Show that on a horizontal plane the range is $R = 2\,a = \frac{v_0^2}{g}\sin 2\epsilon$.

(5) The *time of flight* is the whole time from the beginning of the motion to the instant when the projectile strikes the ground. It is best found by considering the horizontal motion of the projectile alone, which is uniform. Show that on a horizontal plane the time of flight is $T = \frac{2\,v_0}{g}\sin\epsilon$.

(6) Show that the time of flight and the range on a plane through the starting point inclined at an angle θ to the horizon are

$$T_\theta = \frac{2\,v_0}{g}\,\frac{\sin(\epsilon-\theta)}{\cos\theta} \text{ and } R_\theta = \frac{2\,v_0^2}{g}\,\frac{\sin(\epsilon-\theta)\cos\epsilon}{\cos\theta}.$$

(7) What elevation gives the greatest range on a horizontal plane?

(8) Show that on a plane rising at an angle θ to the horizon, to obtain the greatest range, the direction of the initial velocity should bisect the angle between the plane and the vertical.

(9) A stone is dropped from a balloon which, at a height of 1000 ft., is carried along by a horizontal air-current at the rate of 15 miles an hour. (*a*) Where, (*b*) when, and (*c*) with what velocity will it reach the ground?

(10) What must be the initial velocity v_0 of a projectile if with an elevation of 30° it is to strike an object 100 ft. above the horizontal plane of the starting point at a horizontal distance from the latter of 1200 ft.?

(11) What must be the elevation ϵ to strike an object 100 ft. above the horizontal plane of the starting point and 5000 ft. distant, if the initial velocity be 1200 ft. per second?

(12) Show that to strike an object situated in the horizontal plane of the starting point at a distance x from the latter, the elevation must be ϵ or $90° - \epsilon$, where $\epsilon = \frac{1}{2}\sin^{-1}(gx/v_0^2)$.

(13) The initial velocity v_0 being given in magnitude and direction, show how to construct the path graphically.

(14) If it be known that the path of a point is a parabola and that the acceleration is parallel to its axis, show that the acceleration is constant.

(15) Prove that a projectile whose elevation is 60° rises three times as high as when its elevation is 30°, the magnitude of the initial velocity being the same in both cases.

(16) Construct the hodograph for parabolic motion, taking the focus as pole and drawing the radii vectores at right angles to the velocities.

169. A projectile moving in the air or in any other **resisting medium** of uniform density is subject, in addition to the constant acceleration g of gravity, to the resistance of the medium which produces a retardation variable both in magnitude and direction (Art. 125). Experiment shows that this retardation can be expressed in the form cv^n, where c is constant for a given projectile and medium, and n must be determined by experiment for different initial velocities.

170. For $n=1$ the integrations can be readily effected. Resolving the retardation cv into its components $cv_x=cdx/dt$, $cv_y=cdy/dt$, the equations of motion are

$$\frac{d^2x}{dt^2}=-c\frac{dx}{dt}, \quad \frac{d^2y}{dt^2}=-c\frac{dy}{dt}-g. \tag{8}$$

Integrating, we find

$$v_x\equiv\frac{dx}{dt}=v_0\cos\epsilon\cdot e^{-ct}, \quad v_y\equiv\frac{dy}{dt}=\frac{1}{c}[-g+(cv_0\sin\epsilon+g)e^{-ct}], \tag{9}$$

since for $t=0$ we have $v_x=v_0\cos\epsilon$, $v_y=v_0\sin\epsilon$.

The second integration gives

$$x=\frac{v_0}{c}\cos\epsilon(1-e^{-ct}), \quad y=-\frac{g}{c}t+\frac{cv_0\sin\epsilon+g}{c^2}(1-e^{-ct}), \tag{10}$$

since $x=0$, $y=0$ for $t=0$. Eliminating t, we find the equation of the path of the projectile:

$$y=\frac{cv_0\sin\epsilon+g}{v_0\cos\epsilon}\cdot x+\frac{g}{c^2}\log\frac{v_0\cos\epsilon-cx}{v_0\cos\epsilon}. \tag{11}$$

The curve has a vertical asymptote $x=\frac{v_0\cos\epsilon}{c}$; for this value of x, $t=\infty$.

171. Uniform Circular Motion. Let a point P (Fig. 43) describe a circle of radius a with constant angular velocity ω. Its linear velocity $v=\omega a$ is of constant magnitude, but varies in direction. By the formulæ (1), (2) of Art. 159, the tangential acceleration $j_t=0$, while the normal acceleration $j_n=v^2/a=\omega^2 a$ represents the total acceleration. Hence, in uniform circular motion, the acceleration is

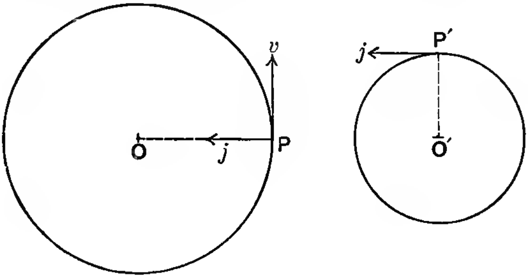

Fig. 43.

$$j=\frac{v^2}{a}=\omega^2 a, \tag{12}$$

and is always directed toward the centre O of the circle.

This appears also by constructing the hodograph of the motion, which is evidently a circle of radius v (Fig. 43). As the acceleration of P is represented by the velocity of the point P' of the hodograph (see Art. 156), we have only to determine this velocity. Let T be the so-called **period**, or **periodic time**, *i.e.* the time in which the point P makes a whole revolution, so that $T=2\pi a/v$; then, since P' describes the circle of radius v in the same time T, we have for the velocity of P' the expression $2\pi v/T$, or substituting for T its value, v^2/a, as above.

172. Simple harmonic motion is a rectilinear motion in which the distance x of the moving point P_x (Fig. 44) from a fixed origin O in the line of motion is a simple harmonic function of the time, *i.e.* a function of the form

$$x=a\cos(\omega t+\epsilon), \text{ or } x=a\sin(\omega t+\epsilon), \tag{13}$$

where a, ω, ϵ are constants.

If the positions P of a point moving uniformly in a circle be projected at every instant on any diameter AA' of the circle, it is easy to see that the motion of the projection P_x along the diameter is simply harmonic. For denoting the constant angular velocity of P by ω, the angle AOP will be $=\omega t$ if the time be counted from the point A. Hence the distance $OP_x=x$ of the point P_x from the centre O, or the **displacement** of P_x at the time t, is

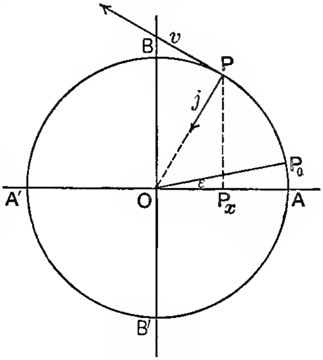

Fig. 44.

$$x=a\cos\omega t,$$

where a is the radius of the circle. This radius $a=OA$ is called the **amplitude** of the simple harmonic motion.

173. While P moves uniformly in the circle, its projection P_x evidently performs oscillations from A through O to A' and back through O to A.

The time T of completing one whole oscillation forward and backward is called the **period** of the simple harmonic motion; it is obviously equal to the period of the motion of P in the circle; *i.e.*

$$T=\frac{2\pi}{\omega}. \tag{14}$$

The period is therefore independent of the amplitude a. It follows that two simple harmonic motions resulting from two uniform circular motions of the same angular velocity on two concentric circles of different radii have the same period; such motions are called **isochronous.**

174. If the time t be counted, not from A, but from some other point P_0 on the circle for which $\measuredangle AOP_0=\epsilon$, we have $\measuredangle AOP=\omega t+\epsilon$, and the equation of the simple harmonic motion is

$$x=a\cos(\omega t+\epsilon). \tag{15}$$

The angle $\omega t+\epsilon$ is called the **phase-angle,** while ϵ is the **epoch-angle,** of the motion. The names *phase* and *epoch* are sometimes applied to these angles, although, strictly speaking, the phase is the *time* (usually expressed as a fraction of the period T) of passing from the position A of maximum displacement to any position P_x, while the epoch is the phase corresponding to the time $t=0$.

175. Differentiating equation (15), we find the velocity

$$v_x\equiv\frac{dx}{dt}=-a\omega\sin(\omega t+\epsilon); \tag{16}$$

and differentiating again, we obtain the acceleration

$$j_x\equiv\frac{d^2x}{dt^2}=-a\omega^2\cos(\omega t+\epsilon)=-\omega^2x \tag{17}$$

of simple harmonic motion.

The same values can be derived by projecting the velocity and acceleration of the uniform circular motion of P on the diameter AA', as is readily seen from Fig. 44.

176. Equation (17) shows that *in simple harmonic motion the acceleration is directly proportional to the distance from the centre.*

Conversely, it can be shown that if the acceleration be proportional to the distance from a fixed point in the direction of the initial velocity, and if it be directed *towards* this point, the motion is simply harmonic. For we then have

$$\frac{d^2x}{dt^2} = -\mu^2 x,$$

where μ is constant. The general integral of this differential equation is (compare Art. 122)

$$x = C_1 \sin \mu t + C_2 \cos \mu t.$$

Differentiating, we find for the velocity

$$v = C_1 \mu \cos \mu t - C_2 \mu \sin \mu t.$$

To determine the constants of integration C_1, C_2, let $s = s_0$ and $v = v_0$ at the time $t = 0$. Substituting these values, we find $s_0 = C_2$ and $v_0 = \mu C_1$; hence

$$x = \frac{v_0}{\mu} \sin \mu t + s_0 \cos \mu t.$$

Putting $v_0/\mu = a \cos \epsilon$, $s_0 = a \sin \epsilon$, which is always allowable, we obtain

$$x = a (\sin \mu t \cos \epsilon + \cos \mu t \sin \epsilon)$$

$$= a \sin (\mu t + \epsilon).$$

This represents a simple harmonic motion whose amplitude is $a = \sqrt{v_0^2 + \mu^2 s_0^2}/\mu$, and whose epoch-angle is $\epsilon = \tan^{-1}(\mu s_0/v_0)$. As the angular velocity of the corresponding uniform circular motion is μ, the period is $T = 2\pi/\mu$.

177. If the uniform circular motion of P be projected on the diameter BB', which is at right angles to the diameter AA' (Fig. 44), we have $OP_y \equiv y = a \sin (\omega t + \epsilon)$. Writing this in the equivalent form

$$y = -a \cos\left(\omega t + \epsilon + \frac{\pi}{2}\right),$$

it appears that the motion of P_y is simply harmonic of the same period and amplitude with the motion of P_x, but differing by $\pi/2$ in phase.

178. Simple harmonic motions occur very frequently in applied mechanics and mathematical physics. A particular case has been treated in Arts. 121–124. As another example we may mention the apparent motion of a satellite about its primary as seen from any point in the plane of the motion, provided the satellite be regarded as moving uniformly in a circle relatively to its primary. Thus the moons of Jupiter, as seen from the earth, have approximately a simple harmonic motion.

179. A mechanism for producing simple harmonic motion can readily be constructed as follows. The end *A* (Fig. 45) of a crank rotating uniformly about the axis *O*, carries a pin running in the slot *AB* of a T bar *ABD* whose axis (produced) passes through the centre *O* of the crank circle. The T bar is constrained by guides to move back and forth along the line *OD*; its motion is evidently simply harmonic, the uniform circular motion of the crank being transformed into rectilinear motion. Compare Art. 151.

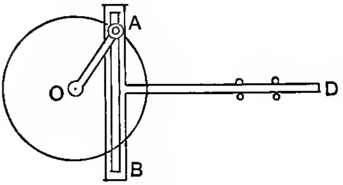

Fig. 45.

180. Exercises.

(1) Show that the maximum acceleration of the simple harmonic motion is numerically equal to the acceleration in the corresponding uniform circular motion.

(2) Find the time of one oscillation from equation (15) without reference to the circular motion.

(3) In the mechanism shown in Fig. 45, if the length of the crank be 2 feet and the number of revolutions 15 per minute, find the velocity and acceleration of the end *D* of the T bar: (*a*) when at elongation; (*b*) when at quarter stroke; (*c*) when at the middle of the stroke.

(4) Show that the period of a simple harmonic oscillation can be expressed in the form $T = 2\pi\sqrt{-x/j_x}$ where j_x is the acceleration of the oscillating point at the time when its distance from the centre, or its displacement, is x.

(5) P_x, P'_x being the positions of the oscillating point at the times t, t', respectively, and δ the angle POP', *i.e.* the difference of phase, show that $t' - t = \delta/\omega$.

(6) Show that $v_x = -\omega\sqrt{a^2 - x^2}$.

181. Compound Harmonic Motion. We have seen (Art. 176) that the motion of a point, whose acceleration is directly proportional to its distance from a fixed centre, and directed towards this centre, is simply harmonic, provided the centre lies in the line of the initial velocity. Removing this last restriction, we have the more general case of compound harmonic motion.

Let O (Fig. 46) be the centre, P the position of the moving point at the time t, $OP = s$ its distance from the centre, v its velocity, $j = -\mu^2 s$ its acceleration, at that time. Referring the motion to two rectangular axes Ox, Oy in the plane determined by v and O, we can resolve v and j into their components along these axes:

$$v_x = v\cos\alpha,\quad v_y = v\sin\alpha,$$

and $j_x = -\mu^2 x$, $j_y = -\mu^2 y$, where α is the angle made by v with the axis Ox, and x, y are the co-ordinates of P.

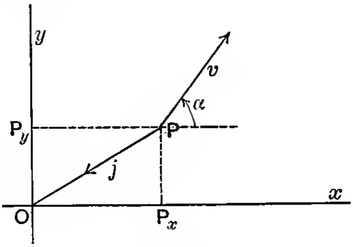

Fig. 46.

The projections P_x, P_y of P on the axes have therefore each a simple harmonic motion, and the motion of P may be regarded as the resultant of these component motions.

182. In general, the motion of P will be curvilinear. We proceed to examine somewhat more in detail the most important cases of the **composition of** two or more **simple harmonic motions,**

beginning with those cases in which the resultant motion is rectilinear.

As, according to Hooke's law, the particles of elastic bodies, after release from strain within the elastic limits, perform small oscillations for which the acceleration is proportional to the displacement from a middle position, the motions under discussion find a wide application in the theories of elasticity, sound, light, and electricity, and form the basis of the general theory of wave motion in an elastic medium.

183. *Two simple harmonic motions in the same line, of equal amplitude* a *and equal period* T, *but differing in phase by* δ, *compound into a simple harmonic motion in the same line, of the same period* T, *but having the amplitude* 2 a *cos* (δ/2).

For we have for the component displacements

$$x_1 = a \cos \omega t, \quad x_2 = a \cos(\omega t + \delta);$$

and as these are in the same line, they can be added algebraically giving the resultant displacement

$$x = x_1 + x_2 = a[\cos \omega t + \cos(\omega t + \delta)],$$

or, by the formula $\cos \alpha + \cos \beta = 2 \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}$,

$$x = 2a \cos \frac{\delta}{2} \cdot \cos\left(\omega t + \frac{\delta}{2}\right).$$

184. *Two simple harmonic motions in the same line, of equal period* T, *but differing both in amplitude and in phase, compound into a single simple harmonic motion in the same line and of the same period.*

For the component displacements

$$x_1 = a_1 \cos(\omega t + \epsilon_1), \quad x_2 = a_2 \cos(\omega t + \epsilon_2)$$

can again be added algebraically, and the resultant displacement is

$$x = x_1 + x_2 = a_1 \cos(\omega t + \epsilon_1) + a_2 \cos(\omega t + \epsilon_2)$$

$$= (a_1 \cos \epsilon_1 + a_2 \cos \epsilon_2) \cos \omega t - (a_1 \sin \epsilon_1 + a_2 \sin \epsilon_2) \sin \omega t.$$

Putting $a_1 \cos \epsilon_1 + a_2 \cos \epsilon_2 = a \cos \epsilon$, $a_1 \sin \epsilon_1 + a_2 \sin \epsilon_2 = a \sin \epsilon$, we have

$$x = a \cos \epsilon \cos \omega t - a \sin \epsilon \sin \omega t$$

$$= a \cos(\omega t + \epsilon),$$

where $a^2 = (a_1 \cos \epsilon_1 + a_2 \cos \epsilon_2)^2 + (a_1 \sin \epsilon_1 + a_2 \sin \epsilon_2)^2$

$$= a_1^2 + a_2^2 + 2\, a_1 a_2 \cos(\epsilon_2 - \epsilon_1)$$

and $\tan \epsilon = (a_1 \sin \epsilon_1 + a_2 \sin \epsilon_2)/(a_1 \cos \epsilon_1 + a_2 \cos \epsilon_2)$.

185. A geometrical illustration of the preceding proposition is obtained by considering the uniform circular motions corresponding to the simple harmonic motions (Fig. 47).

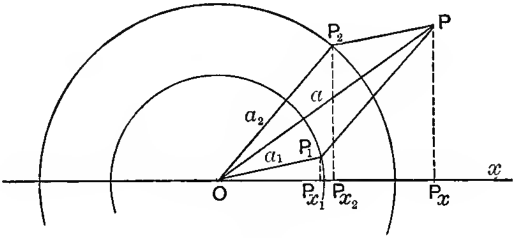

Fig. 47.

Drawing the radii $OP_1 = a_1$, $OP_2 = a_2$ so as to include an angle equal to the difference of phase $\epsilon_2 - \epsilon_1$ and completing the parallelogram OP_1PP_2, it appears from the figure that the diagonal OP of this parallelogram represents the resulting amplitude a. For since P_1P is equal and parallel to OP_2, we have for the projections on Ox the relation $OP_{x_1} + OP_{x_2} = OP_x$, or $x_1 + x_2 = x$.

Again, if the angle xOP_1 be taken equal to the epoch-angle ϵ_1, and hence $xOP_2 = \epsilon_2$, the angle xOP represents the epoch ϵ of the resulting motion.

We thus have a simple geometrical construction for the elements a, ϵ of the resulting motion from the elements a_1, ϵ_1 and a_2, ϵ_2 of the component motions. As the period is the same for the two component motions, the points P_1 and P_2 describe their respective circles with equal angular velocity so that the parallelogram OP_1PP_2 does not change its form in the course of the motion.

186. The construction given in the preceding article can be described briefly by saying that two simple harmonic motions of equal period in the same line are compounded by *geometrically adding* their amplitudes, it being understood that the phase-angles determine the directions in which the amplitudes are to be drawn.

It follows at once that not only two, but *any number of simple harmonic motions, of equal period in the same line, can be compounded by geometric addition of their amplitudes into a single simple harmonic motion in the same line and of the same period.*

Conversely, any given simple harmonic motion can be resolved into two or more components in the same line and of the same period.

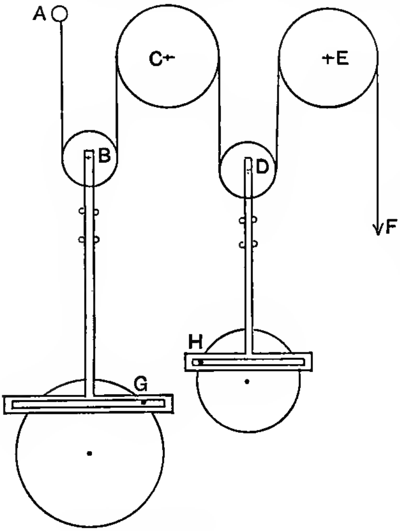

Fig. 48.

187. The kinematical meaning of this composition of simple harmonic motions of equal period in the same line will perhaps be best understood from the mechanism sketched in Fig. 48. A cord runs from the fixed point A over the movable pulleys B, D and the fixed pulleys C, E, and ends in F. Each of the movable pulleys receives a vertical simple harmonic motion from the T bars *BG* and *DH*, just as in Fig. 45 (Art. 179). If the free end *F* of the cord be just kept tight, its vertical displacement will be twice the sum of the vertical displacements of *B* and *D*, and as these points have simple harmonic motions, the motion of *F* will be twice the resultant simple harmonic motion.

The idea of this mechanism is due to Lord Kelvin.

188. Exercises.

(1) Find the resultant of three simple harmonic motions in the same line, and all of period $T = 10$ seconds, the amplitudes being 5, 3, and 4 cm., and the phase differences 30° and 60°, respectively, between the first and second, and the first and third motions.

(2) Apply the geometrical method of Art. 185 to the problem of Art. 183.

(3) Find the resultant of two simple harmonic motions in the same line and of equal period when the amplitudes are equal and the phases differ: (*a*) by an even multiple of π, (*b*) by an odd multiple of π.

(4) Resolve $x = 10 \cos(\pi t + 45°)$ into two components in the same line with a phase difference of 30°, one of the components having the epoch 0.

(5) Trace the curves representing the component motions as well as the resultant motion in Ex. (1), taking the time as abscissa and the displacement as ordinate.

(6) Show that the resultant of n simple harmonic motions of equal period T in the same line, viz.:

$$x_1 = a_1 \cos\left(\frac{2\pi}{T}t + \epsilon_1\right),\ x_2 = a_2 \cos\left(\frac{2\pi}{T}t + \epsilon_2\right),\ \cdots\ x_n = a_n \cos\left(\frac{2\pi}{T}t + \epsilon_n\right),$$

is the isochronous simple harmonic motion

$$x = a \cos\left(\frac{2\pi}{T}t + \epsilon\right),$$

where
$$a^2 = \left(\sum_1^n a_i \cos \epsilon_i\right)^2 + \left(\sum_1^n a_i \sin \epsilon_i\right)^2$$

and
$$\tan \epsilon = \sum_1^n a_i \sin \epsilon_i \Big/ \sum_1^n a_i \cos \epsilon_i.$$

189. The composition of two or more simple harmonic motions in the same line can readily be effected, even *when the components differ in period. But the resultant motion is not simply harmonic.*

Thus, for two components

$$x_1 = a_1 \cos(\omega_1 t + \epsilon_1), \quad x_2 = a_2 \cos(\omega_2 t + \epsilon_2),$$

putting $\omega_2 t + \epsilon_2 = \omega_1 t + (\omega_2 - \omega_1) t + \epsilon_2 = \omega_1 t + \epsilon_1 + \delta$, say, where $\delta = (\omega_2 - \omega_1) t + \epsilon_2 - \epsilon_1$ is the difference of phase at the time t, we have for the resulting motion

$$x = x_1 + x_2 = a_1 \cos(\omega_1 t + \epsilon_1) + a_2 \cos(\omega_1 t + \epsilon_1 + \delta);$$

and treating this similarly as in Art. 184 we find

$$x = (a_1 + a_2 \cos\delta) \cos(\omega_1 t + \epsilon_1) - a_2 \sin\delta \sin(\omega_1 t + \epsilon_1),$$

or putting $a_1 + a_2 \cos\delta = a \cos\epsilon, \quad a_2 \sin\delta = a \sin\epsilon$,

$$x = a \cos(\omega_1 t + \epsilon_1 + \epsilon),$$

where $a^2 = a_1^2 + a_2^2 + 2 a_1 a_2 \cos\delta$ and $\tan\epsilon = a_2 \sin\delta / (a_1 + a_2 \cos\delta)$.

190. These formulæ can be interpreted geometrically by Fig. 47, similarly as in Art. 185. But as in the present case the angle δ, and consequently the quantities a and ϵ in the expression for x, are variable, the parallelogram OP_1PP_2 while having constant sides has variable angles and changes its form in the course of the motion.

A mechanism similar to that of Fig. 48 (Art. 187), can be used to effect mechanically the composition of simple harmonic motions in the same line whether the periods be equal or not. This is the principle of the tide-predicting machine devised by Lord Kelvin.*

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* See THOMSON and TAIT, *Natural philosophy*, Vol. I., Part I., new edition, 1879, p. 43 sq. and p. 479 sq. and J. D. EVERETT, *Vibratory motion and sound*, 1882.

191. To show the connection of the present subject with the theory of **wave motion**, imagine a flexible cord *AB* of which one end *B* is fixed while the other *A* is given a sudden jerk or transverse motion from *A* to *C* and back through *A* to *D*, etc. (Fig. 49). The displacement given to *A* will, so to speak, run along the cord, travelling from *A* to *B* and producing a wave. The figure exhibits the successive stages of the motion up to the time when a complete wave has been produced.

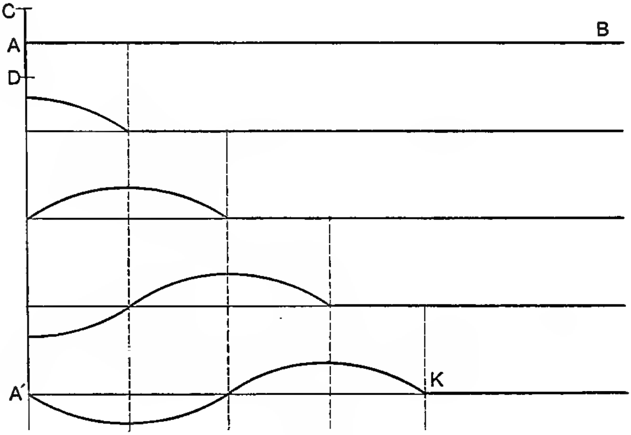

Fig. 49.

192. The distance $A'K$ (Fig. 49) is called the **length of the wave.** Denoting this length by λ, and the time in which the motion spreads from A' to K by T we have for the **velocity of propagation** of the wave

$$V=\frac{\lambda}{T}. \tag{18}$$

It is to be noticed that the motion of any particular point of the cord is supposed to be rectilinear and at right angles to *AB*; this is the case with the simple transverse vibrations in an elastic medium such as the luminiferous ether regarded as the vehicle of light waves.

193. If the motion of A be simply harmonic, say $y=a\sin\omega t$, the motions of the successive points of the cord will differ from the motion of A only in phase, and the displacements of all these points at any time t can be represented by

$$y=a\sin(\omega t-\epsilon), \tag{19}$$

where ϵ varies from 0 to 2π as we pass from A to K.

As the time T in which the motion spreads from A to K is equal to the period of a vibration of A (or of any other point of the cord), we have $\omega=2\pi/T$, or, by (18), $\omega=2\pi V/\lambda$. And if x be the distance of the point of the cord under consideration from A, we must have $x:\lambda=\epsilon:2\pi$; that is, $\epsilon=2\pi x/\lambda$. Substituting these values of ω and ϵ in (19), the equation of the wave motion can be written in the form

$$y=a\sin\frac{2\pi}{\lambda}(Vt-x). \tag{20}$$

194. This equation can be looked upon from two different points of view according as we regard t or x as variable.

Let t be constant; *i.e.* let us consider the displacements of all points of the cord at a given instant. If for x in (20) we substitute $x+n\lambda$, where n is any positive or negative integer, the angle $(Vt-x)\,2\pi/\lambda$ is changed by $2\pi n$, so that the value of y remains unchanged. The displacements of all particles whose distances from A differ by whole wave lengths are therefore the same; in other words, the state of motion at any instant is represented by a series of equal waves.

Now let x be constant, and t variable. Substituting for t in (20) the value $t+nT=t+n\lambda/V$, the angle $(Vt-x)\,2\pi/\lambda$ is again changed by $2\pi n$, and y remains the same. This shows the periodicity in the motion of any given particle.

195. If the point A (Fig. 49) be subjected simultaneously to more than one simple harmonic motion, the displacements resulting from each must be added algebraically, thus forming a compound wave which can readily be traced by first tracing the component waves and then adding their ordinates, or analytically by forming the equation of the resultant motion as in Art. 189.

196. Exercise.

(1) Trace the wave produced by the superposition of two simple harmonic motions in the same line of equal amplitudes, the periods being as 2 : 1, (*a*) when they do not differ in phase, (*b*) when their epochs differ by 7/16 of the period.

197. The idea of wave motion implies that the displacement y should be a periodic function of x and t such as to fulfil the following conditions: y must assume the same value (*a*) when x is changed into $n\lambda$, (*b*) when t is changed into $t+T$, (*c*) when both changes are made simultaneously; the constants λ and T being connected by the relation $\lambda = VT$.

The condition (*c*) requires y to be of the form $y=f(Vt-x)$; for $Vt-x$ remains unchanged when x is replaced by $x+VT$ and at the same time t by $t+T$.

A particular case of such a function is $y=a\sin c(Vt-x)$. As y should remain unchanged when t is replaced by $t+T$, we must have $c=2\pi/VT=2\pi/\lambda$. Thus the function

$$y=a\sin\frac{2\pi}{\lambda}(Vt-x)$$

fulfils the three conditions (*a*), (*b*), (*c*). Putting as before (Art. 193) $2\pi x/\lambda=-\epsilon$, we can write it

$$y=a\sin\left(\frac{2\pi}{T}t+\epsilon\right).$$

198. The importance of this particular solution of our problem lies in the fact that, according to **Fourier's theorem**, any single-valued periodic function of period T can be expanded, between definite limits of the variable, into a series of the form

$$f(t)=a_0+a_1\sin\left(\frac{2\pi}{T}\cdot t+\epsilon_1\right)+a_2\sin\left(\frac{2\pi}{T}\cdot 2t+\epsilon_2\right)$$
$$+a_3\sin\left(\frac{2\pi}{T}\cdot 3t+\epsilon_3\right)+\cdots. \qquad (21)$$

As applied to the theory of wave motion this means of course that any wave motion, however complex, can be regarded as made up of a series of superposed simple harmonic vibrations of periods T, $T/2$, $T/3$, $\cdots$, or since $T = \lambda/V$, of wave lengths λ, $\lambda/2$, $\lambda/3$, $\cdots$.

199. A full discussion of Fourier's theorem cannot be given in this place.* We wish, however, to show its practical application in an example.

The equation (21) can be written in the form

$$f(t) = a_1 \cos \epsilon_1 \sin \frac{2\pi}{T} \cdot t + a_2 \cos \epsilon_2 \sin \frac{2\pi}{T} \cdot 2t + a_3 \cos \epsilon_3 \sin \frac{2\pi}{T} \cdot 3t + \cdots$$

$$+ a_0 + a_1 \sin \epsilon_1 \cos \frac{2\pi}{T} \cdot t + a_2 \sin \epsilon_2 \cos \frac{2\pi}{T} \cdot 2t + a_3 \sin \epsilon_3 \cos \frac{2\pi}{T} \cdot 3t + \cdots,$$

or putting $2\pi t/T = x$, $a_1 \sin \epsilon_1 = A_1$, $a_2 \sin \epsilon_2 = A_2$, $\cdots$, $a_1 \cos \epsilon_1 = B_1$, $a_2 \cos \epsilon_2 = B_2$, $\cdots$,

$$f(x) = a_0 + A_1 \cos x + A_2 \cos 2x + A_3 \cos 3x + \cdots$$
$$+ B_1 \sin x + B_2 \sin 2x + B_3 \sin 3x + \cdots. \qquad (22)$$

This is known as Fourier's series. According to the nature of the function to be expanded, it is often sufficient to use the sine series or the cosine series alone. As the method of determining the coefficients is always the same, it will be sufficient to consider the simple sine series:

$$f(x) = B_1 \sin x + B_2 \sin 2x + B_3 \sin 3x + \cdots. \qquad (23)$$

200. The problem before us can now be stated as follows: Given any single-valued function of x, either by its analytical expression or by the trace of the curve representing it, to determine the coefficients B in (23) so as to make the right-hand member of this equation represent the values of the given function between certain finite limits of x.

We shall assume these limits to be $x = 0$ and $x = \pi$; and we shall first try to make $n - 1$ points of the given curve taken between these limits coincide with as many points of the curve

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* The student is referred to THOMSON and TAIT, *Natural philosophy*, I. 1, 1879, pp. 55–60; also to B. RIEMANN, *Partielle Differentialgleichungen*, herausgegeben von K. Hattendorff, 3d ed., Braunschweig, Vieweg, 1882, pp. 44–95, and to G. M. MINCHIN, *Uniplanar kinematics*, Oxford, Clarendon Press, 1882, p. 13 sq.

$$f(x) = B_1 \sin x + B_2 \sin 2x + \cdots + B_{n-1} \sin (n - 1)x. \qquad (24)$$

Then passing to the limit for $n = \infty$, the problem will be solved.

201. Dividing the interval from $x = 0$ to $x = \pi$ into n equal parts and taking the $n - 1$ points on the given curve so that their abscissæ are

$$\frac{\pi}{n},\ \frac{2\pi}{n},\ \frac{3\pi}{n},\ \cdots\ \frac{(n-1)\pi}{n},$$

the curve (24) will contain these points if the following $n - 1$ equations are fulfilled:

$$\left.\begin{array}{llll}
f\left(\frac{\pi}{n}\right) = B_1 \sin 1 \cdot \frac{\pi}{n} & + B_2 \sin 1 \cdot \frac{2\pi}{n} & + \cdots + B_{n-1} \sin 1 \cdot \frac{(n-1)\pi}{n}, \\
f\left(\frac{2\pi}{n}\right) = B_1 \sin 2 \cdot \frac{\pi}{n} & + B_2 \sin 2 \cdot \frac{2\pi}{n} & + \cdots + B_{n-1} \sin 2 \cdot \frac{(n-1)\pi}{n}, \\
f\left(\frac{3\pi}{n}\right) = B_1 \sin 3 \cdot \frac{\pi}{n} & + B_2 \sin 3 \cdot \frac{2\pi}{n} & + \cdots + B_{n-1} \sin 3 \cdot \frac{(n-1)\pi}{n}, \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
f\left(\frac{(n-1)\pi}{n}\right) = B_1 \sin (n-1) \frac{\pi}{n} & + B_2 \sin (n-1) \frac{2\pi}{n} & + \cdots + B_{n-1} \sin (n-1) \frac{(n-1)\pi}{n}.
\end{array}\right\} \quad \cdots \quad (25)$$

These equations are sufficient to determine the $n - 1$ unknown constants $B_1, B_2, \cdots, B_{n-1}$.

202. To solve the equations we multiply them by indeterminate coefficients and add. The coefficients can be so selected as to make all the unknown quantities disappear except one which is thus determined.

Thus, to find B_m multiply each equation by twice the coefficient of B_m in this equation, viz. the first equation by $2 \sin (m\pi/n)$, the second by $2 \sin (2\,m\pi/n)$, etc., the last by $2 \sin [(n-1)m\pi/n]$.

After adding, the factor of B_k will be

$$\beta_k = 2\left[\sin k \frac{\pi}{n} \sin m \frac{\pi}{n} + \sin k \frac{2\pi}{n} \sin m \frac{2\pi}{n} + \cdots \right.$$
$$\left. + \sin k (n-1) \frac{\pi}{n} \sin m (n-1) \frac{\pi}{n}\right].$$

Transforming every term by the formula $2\sin\alpha\sin\beta=\cos(\alpha-\beta)-\cos(\alpha+\beta)$, we find

$$\beta_k=\cos(k-m)\frac{\pi}{n}+\cos(k-m)\frac{2\pi}{n}+\cdots+\cos(k-m)\frac{(n-1)\pi}{n}$$
$$-\left[\cos(k+m)\frac{\pi}{n}+\cos(k+m)\frac{2\pi}{n}+\cdots+\cos(k+m)\frac{(n-1)\pi}{n}\right]. \quad . \quad . \quad . \quad (26)$$

Applying the trigonometrical formula

$$\cos\alpha+\cos 2\alpha+\cdots+\cos(n-1)\alpha=\frac{1}{2}\left(-1+\frac{\sin(2n-1)\alpha/2}{\sin\alpha/2}\right),$$

we obtain

$$2\beta_k=\frac{\sin(2n-1)(k-m)\frac{\pi}{2n}}{\sin(k-m)\frac{\pi}{2n}}-\frac{\sin(2n-1)(k+m)\frac{\pi}{2n}}{\sin(k+m)\frac{\pi}{2n}}$$

$$=\frac{\sin(k-m)\pi\cos(k-m)\frac{\pi}{2n}-\cos(k-m)\pi\sin(k-m)\frac{\pi}{2n}}{\sin(k-m)\frac{\pi}{2n}}$$

$$-\frac{\sin(k+m)\pi\cos(k+m)\frac{\pi}{2n}-\cos(k+m)\pi\sin(k+m)\frac{\pi}{2n}}{\sin(k+m)\frac{\pi}{2n}}.$$

If k be different from m, this reduces to

$$\beta_k=\tfrac{1}{2}[-\cos(k-m)\pi+\cos(k+m)\pi],$$

and this is always $=0$, since $k+m$ and $k-m$ are either both odd or both even.

If $k=m$, we find from (26)

$$\beta_m=1+1+\cdots+1$$
$$-\left[\cos 2m\frac{\pi}{n}+\cos 2m\frac{2\pi}{n}+\cdots+\cos 2m\frac{(n-1)\pi}{n}\right]$$

$$= n - 1 - \frac{1}{2}\left[-1 + \frac{\sin(2n-1)\dfrac{m\pi}{n}}{\sin\dfrac{m\pi}{n}}\right]$$

$$= n - \frac{1}{2} - \frac{1}{2}\,\frac{\sin 2m\pi\cos\dfrac{m\pi}{n} - \cos 2m\pi\sin\dfrac{m\pi}{n}}{\sin\dfrac{m\pi}{n}}$$

$$= n - \tfrac{1}{2} - \tfrac{1}{2}(-1) = n.$$

We have, therefore, finally

$$nB_m = 2f\left(\frac{\pi}{n}\right)\sin m\frac{\pi}{n} + 2f\left(\frac{2\pi}{n}\right)\sin m\frac{2\pi}{n} + \cdots$$
$$+ 2f\left(\frac{(n-1)\pi}{n}\right)\sin m\frac{(n-1)\pi}{n}, \qquad (27)$$

with $m = 1, 2, 3, \cdots n-1$.

203. It remains to pass to the limit when $n = \infty$ and $\frac{\pi}{n}$ vanishes. Writing (27) in the form

$$B_m = \frac{2}{\pi}\cdot\frac{\pi}{n}\left[\left(\frac{\pi}{n}\right)\sin m\frac{\pi}{n} + f\left(\frac{2\pi}{n}\right)\sin m\frac{2\pi}{n} + \cdots\right.$$
$$\left. + f\left(\frac{(n-1)\pi}{n}\right)\sin m\frac{(n-1)\pi}{n}\right],$$

we obviously have in the limit

$$B_m = \frac{2}{\pi}\int_0^\pi f(x)\sin mx\,dx, \quad m = 1, 2, 3, \cdots. \qquad (28)$$

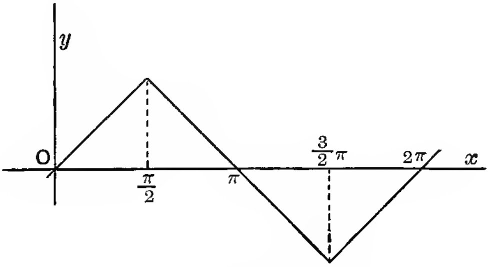

Fig. 50.

204. As an application let us determine the series representing the broken line formed by the two sides of an isosceles right-angled triangle whose hypotenuse lies in the axis of x (Fig. 50).

We assume the length of this hypotenuse $=\pi$; then the given function is $f(x)=x$ from $x=0$ to $x=\pi/2$, and $f(x)=\pi-x$ from $x=\pi/2$ to $x=\pi$.

On account of the discontinuity at the point $x=\pi/2$, the integral in (28) must be resolved into two, and we have

$$B_m=\frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} x\sin mx\,dx+\int_{\frac{\pi}{2}}^{\pi}(\pi-x)\sin mx\,dx\right]$$

$$=\frac{2}{\pi}\left[-\frac{1}{m}\frac{\pi}{2}\cos\frac{m\pi}{2}+\frac{1}{m^2}\sin\frac{m\pi}{2}+\frac{1}{m}\frac{\pi}{2}\cos\frac{m\pi}{2}+\frac{1}{m^2}\sin\frac{m\pi}{2}\right]$$

$$=\frac{4}{\pi m^2}\sin\frac{m\pi}{2}.$$

For even values of m, $\sin(m\pi/2)=0$; for odd values, $\sin(m\pi/2)$ is alternately positive or negative. Hence the series (23) becomes

$$f(x)=\frac{4}{\pi}\left[\frac{\sin x}{1^2}-\frac{\sin 3x}{3^2}+\frac{\sin 5x}{5^2}-\dots\right]. \tag{29}$$

This expansion certainly holds when x lies between 0 and π. As every term of the series vanishes for $x=0$ as well as for $x=\pi$, the expansion holds even at these limits. Moreover, when x lies between π and 2π, all the terms of the series, with signs reversed, pass through the same succession of values as between 0 and π. The series represents therefore between these limits an equal triangle with its vertex below the axis of x (Fig. 50). Beyond the point $x=2\pi$, the same figure repeats itself owing to the periodicity of the sine.

It thus appears that the series represents an infinite zigzag line for all values of x.

205. We proceed to the *composition of simple harmonic motions not in the same line.* We shall, however, assume that all the component motions lie in the same plane.

It is evident that the projection of a simple harmonic motion on any line is again a simple harmonic motion of the same period and phase and with an amplitude equal to the projection of the original amplitude.

Hence, to compound any number of simple harmonic motions along lines lying in the same plane, we may project all these motions on any two rectangular axes Ox, Oy taken in this plane, and compound, by Art. 184 or 189, the components lying in the same axis. It then only remains to compound the two motions, one along Ox, the other along Oy, into a single motion.

206. Just as in Arts. 184, 189, we must distinguish two cases: (a) when the given motions have all the same period, and (b) when they have not.

In the former case, by Art. 184, the two components along Ox and Oy will have equal periods, *i.e.* they will be of the form

$$x = a \sin \omega t, \quad y = b \sin(\omega t + \delta). \tag{30}$$

The path of the resulting motion is obtained by eliminating t between these equations. We have

$$\frac{y}{b} = \sin \omega t \cos \delta + \cos \omega t \sin \delta$$

$$= \frac{x}{a} \cos \delta + \sqrt{1 - \frac{x^2}{a^2}} \sin \delta.$$

Writing this equation in the form

$$\left(\frac{y}{b} - \frac{x}{a} \cos \delta\right)^2 = \left(1 - \frac{x^2}{a^2}\right) \sin^2 \delta,$$

or

$$\frac{x^2}{a^2} - \frac{2xy}{ab} \cos \delta + \frac{y^2}{b^2} = \sin^2 \delta, \tag{31}$$

we see that it represents an ellipse (since $\frac{1}{a^2} \cdot \frac{1}{b^2} - \frac{\cos^2 \delta}{a^2 b^2} = \left(\frac{\sin \delta}{ab}\right)^2$ is positive) whose centre is at the origin. The resultant motion is therefore called **elliptic harmonic motion.**

207. Although in what precedes we have assumed the axes at right angles to each other, this is not essential. The same equation (31) is obtained for oblique axes Ox, Oy, and it is easy to show (say by transforming (31) to rectangular axes) that this equation still represents an ellipse. We have, therefore, the general result that *any number of simple harmonic motions of the same period and in the same plane, whatever may be their directions, amplitudes, and phases, compound into a single elliptic harmonic motion.*

208. A few particular cases may be noticed. The equation (31) will represent a (double) straight line, and hence the elliptic vibration will degenerate into a simple harmonic vibration, whenever $\sin^2\delta=0$, *i.e.* when $\delta=n\pi$, where n is a positive or negative integer. In this case $\cos\delta$ is $+1$ or -1, and (31) reduces to

$$\frac{x}{a}-\frac{y}{b}=0, \text{ if } \delta=2n\pi,$$

and to

$$\frac{x}{a}+\frac{y}{b}=0, \text{ if } \delta=(2n+1)\pi.$$

Thus two rectangular vibrations of the same period compound into a simple harmonic vibration when they differ in phase by an integral multiple of π, that is when one lags behind the other by half a wave length.

209. Again, the ellipse (31) reduces to a circle only when $\cos\delta=0$, *i.e.* $\delta=(2n+1)\pi/2$, and in addition $a=b$, the co-ordinates being assumed orthogonal.

Thus two rectangular vibrations of equal period and amplitude compound into a circular vibration if they differ in phase by $\pi/2$, *i.e.* if one is retarded behind the other by a quarter of a wave length.

This circular harmonic motion is evidently nothing but uniform motion in a circle; and we have seen in Art. 172 that, conversely, uniform circular motion can be resolved into two rectangular simple harmonic vibrations of equal period and amplitude, but differing in phase by $\pi/2$.

The results of Arts. 205–209 can also be established by purely geometrical methods of an elementary character.*

210. It remains to consider the case when the given simple harmonic motions do not all have the same period. It follows from Art. 189 that in this case, if we again project the given motions on two rectangular axes *Ox*, *Oy*, the resulting motions along *Ox*, *Oy* are in general not simply harmonic.

The elimination of t between the expressions for x and y may present difficulties. But, of course, the curve can always be traced by points, graphically.

We shall here consider only the case when the motions along *Ox* and *Oy* are simply harmonic.

211. *If two simple harmonic motions along the rectangular directions* Ox, Oy, viz.:

$$x = a_1 \cos\left(\frac{2\pi}{T_1}t + \epsilon_1\right), \quad y = a_2 \cos\left(\frac{2\pi}{T_2}t + \epsilon_2\right),$$

of different amplitudes, phases, and *periods* are to be compounded, the resulting motion will be confined within a rectangle whose sides are $2a_1$, $2a_2$, since these are the maximum values of $2x$ and $2y$.

The path of the moving point will be a *closed* curve only when the quotient T_1/T_2 is a commensurable number, say $= m/n$, where m is prime to n. The x co-ordinate of the curve will have m maxima, the y co-ordinate n, and the whole curve will be traversed after m vibrations along *Ox* and n along *Oy*.

The formation of the resulting curve will best be understood from the following example.

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* See, for instance, J. G. MACGREGOR, *An elementary treatise on kinematics and dynamics*, London, Macmillan, 1887, pp. 115 sq.

212. Let $a_1=a_2=a$, $\epsilon_1=0$, $\epsilon_2=\delta$, and let the ratio of the periods be $T_1/T_2=2/1$. The equations of the component simple harmonic vibrations are

$$x=a\cos\omega t,\ y=a\cos(2\omega t+\delta).$$

Here it is easy to eliminate t. We have

$$y=a\cos 2\omega t\cos\delta-a\sin 2\omega t\sin\delta$$

$$=a\left(2\frac{x^2}{a^2}-1\right)\cos\delta-2a\frac{x}{a}\sqrt{1-\frac{x^2}{a^2}}\sin\delta.$$

Hence the equation of the path is:

$$ay=(2x^2-a^2)\cos\delta-2x\sqrt{a^2-x^2}\sin\delta.$$

If there be no difference of phase between the components, *i.e.* if $\delta=0$, this reduces to the equation of a parabola:

$$x^2=\frac{a}{2}(y+a).$$

For $\delta=\pi/2$, the equation also assumes a simple form:

$$a^2y^2=4x^2(a^2-x^2).$$

213. It is instructive to trace the resulting curves for a given ratio of periods and for a series of successive differences of phase (*Lissajous's Curves*).

Thus, in Fig. 51, the curve for $T_1/T_2=3/4$, and for a phase difference $\delta=0$ is the fully drawn curve, while the dotted curve represents the path for the same ratio of the periods when the phase difference is one-twelfth of the smaller period. The equations of the components are for the full curve

$$x=6\cos\frac{2\pi}{3}t,\ y=5\cos\frac{2\pi}{4}t,$$

and for the dotted curve

$$x-6\cos\left(\frac{2\pi}{3}t+\frac{2\pi}{12}\right),\ y=5\cos\frac{2\pi}{4}t.$$

In tracing these curves, imagine the simple harmonic motions replaced by the corresponding uniform circular motions (Fig. 51). With the amplitudes 6, 5 as radii, describe the semicircles *ADB*, *AEC*, so that *BC* is the rectangle within which the curves are confined; the intersection of the diagonals of this rectangle is the origin *O*, *AB* is parallel to the axis of x, *AC* to the axis of y. Next divide the circles over *AB*, *AC* into parts corresponding to equal intervals of time. In the present case, the periods for *AB*, *AC* being as 3 to 4, the circle over *AB* must be divided into $3n$ equal parts, that over *AC* into $4n$. In the figure, n is taken as 4, the circles being divided into 12 and 16 equal parts, respectively.

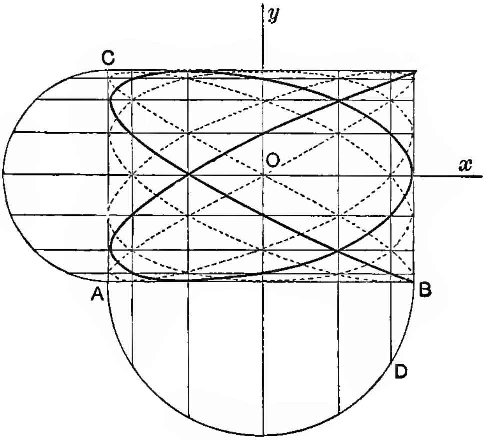

Fig. 51.

The first point of the full drawn curve corresponds to $t=0$, that is $x=6, y=5$; this gives the upper right hand corner of the rectangle. The next point is the intersection of the vertical line through *D* and the horizontal line through *E*, the arcs $BD=1/12$ of the circle over *AB*, and $CE=1/16$ of that over *AC* being described in the same time, so that the co-ordinates of the corresponding point are

$$x=6\cos\left(\frac{2\pi}{3}\cdot\frac{3}{12}\right)=6\cos\left(2\pi\cdot\frac{1}{12}\right)$$

$$y=5\cos\left(\frac{2\pi}{4}\cdot\frac{4}{16}\right)=5\cos\left(2\pi\cdot\frac{1}{16}\right).$$

Similarly the next point

$$x=6\cos\left(2\pi\cdot\frac{2}{12}\right),\ y=5\cos\left(2\pi\cdot\frac{2}{16}\right)$$

is found from the next two points of division on the circles, etc.

To construct the dotted curve, it is only necessary to begin on the circle over AB with D as first point of division.

214. Exercises.

(1) With the data of Art. 213 construct the curves for phase differences of 2/12, 3/12, ⋯ 11/12 of the smaller period.

(2) Construct the curves (Art. 212)

$$x=a\cos\omega t,\ y=a\cos(2\omega t+\delta)$$

for $\delta=0,\ \pi/4,\ \pi/2,\ 3\pi/4,\ \pi,\ 5\pi/4,\ 3\pi/2,\ 7\pi/4,\ 2\pi$.

(3) Trace the path of a point subjected to two circular vibrations of the same amplitude, but differing in period, (*a*) when the sense is the same; (*b*) when it is opposite.

215. The **mathematical pendulum** is a point compelled to move in a vertical circle under the acceleration of gravity.

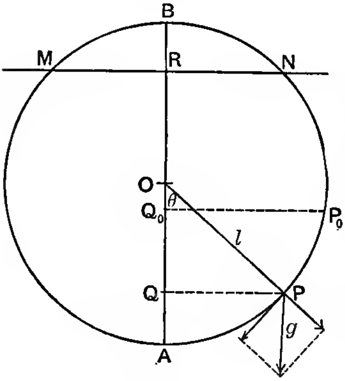

Fig. 52.

Let O be the centre (Fig. 52), A the lowest, and B the highest point of the circle. The radius $OA=l$ of the circle is called the length of the pendulum. Any position P of the moving point is determined by the angle $AOP=\theta$ counted from the vertical radius OA in the positive (counterclockwise) sense of rotation.

If P_0 be the initial position of the moving point at the time

$t=0$, and $\measuredangle\, AOP_0=\theta_0$, then the arc $P_0P=s$ described in the time t is $s=l(\theta_0-\theta)$; hence $v=ds/dt=-ld\theta/dt$, and $dv/dt=-ld^2\theta/dt^2$, the negative sign indicating that θ diminishes as s and t increase.

Resolving the acceleration of gravity, g, into its normal and tangential components $g\cos\theta$, $g\sin\theta$, and considering that the former is without effect owing to the condition that the point is constrained to move in a circle, we obtain the equation of motion in the form $dv/dt=g\sin\theta$, or

$$l\frac{d^2\theta}{dt^2}+g\sin\theta=0. \tag{32}$$

216. The first integration is readily performed by multiplying the equation by $d\theta/dt$ which makes the left-hand member an exact derivative,

$$\frac{d}{dt}\left(\frac{l}{2}\left(\frac{d\theta}{dt}\right)^2-g\cos\theta\right);$$

hence integrating, we obtain

$$\tfrac{1}{2}l\left(\frac{d\theta}{dt}\right)^2-g\cos\theta=C,$$

or considering that $v=-ld\theta/dt$,

$$\tfrac{1}{2}v^2-gl\cos\theta=Cl.$$

To determine the constant C, the initial velocity v_0 at the time $t=0$ must be given. We then have $\frac{1}{2}v_0^2-gl\cos\theta_0=Cl$; hence

$$\tfrac{1}{2}v^2=\tfrac{1}{2}v_0^2-gl\cos\theta_0+gl\cos\theta=g\left(\frac{v_0^2}{2g}-l\cos\theta_0+l\cos\theta\right). \tag{33}$$

The right-hand member can readily be interpreted geometrically; $v_0^2/2g$ is the height by falling through which the point would acquire the initial velocity v_0 (see Art. 113); $l\cos\theta$

$-l\cos\theta_0 = OQ - OQ_0 = QQ_0$, if Q, Q_0 are the projections of P, P_0 on the vertical AB. If we draw a horizontal line MN at the height $v_0^2/2g$ above P_0 and if this line intersect the vertical AB in R, we have for the velocity v the expression:

$$\tfrac{1}{2}v^2 = g\cdot QR. \tag{34}$$

If the initial velocity be $=0$, the equation would be

$$\tfrac{1}{2}v^2 = g\cdot QQ_0. \tag{35}$$

At the points M, N where the horizontal line MN intersects the circle the velocity becomes 0. The point can therefore never rise above these points.

Now, according to the value of the initial velocity v_0, the line MN may intersect the circle in two real points M, N, or touch it at B, or not meet it at all. In the first case the point P performs oscillations, passing from its initial position P_0 through A up to M, then falling back to A and rising to N, etc. In the third case P makes complete revolutions.

217. The second integration of the equation of motion cannot be effected in finite terms, without introducing elliptic functions. But for the case of most practical importance, viz. for very small values of θ, it is easy to obtain an approximate solution. In this case θ can be substituted for $\sin\theta$, and the equation becomes:

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0. \tag{36}$$

This is a well known differential equation (compare Art. 122, Eq. (19), and Art. 176), whose general integral is

$$\theta = C_1 \cos t\sqrt{\frac{g}{l}} + C_2 \sin t\sqrt{\frac{g}{l}}.$$

The constants C_1, C_2 can be determined from the initial conditions for which we shall now take $\theta=\theta_0$ and $v=0$ when $t=0$; this gives $C_1=\theta_0$, $C_2=0$; hence

$$\theta=\theta_0 \cos t\sqrt{\frac{g}{l}}, \quad t=\sqrt{\frac{l}{g}}\cos^{-1}\frac{\theta}{\theta_0}. \tag{37}$$

The last equation gives for $\theta=-\theta_0$ the time t_1 of one oscillation, or half the period T,

$$t_1=\tfrac{1}{2}T=\pi\sqrt{\frac{l}{g}}. \tag{38}$$

The time of a small oscillation is thus seen to be independent of the arc through which the pendulum swings; in other words, for all small arcs the times of oscillation of the same pendulum are the same; such oscillations are therefore called *isochronous*.

218. A pendulum whose length is so adjusted as to make it perform at a certain place just one oscillation in a second is called a **seconds pendulum.**

Putting $t_1=1$ in (38) we find for the length l_1 of the seconds pendulum at a place where the acceleration of gravity is g,

$$l_1=\frac{g}{\pi^2}. \tag{39}$$

As the length of the pendulum can be determined with great accuracy by measurement, the pendulum can be used to find the value of g.

The length of the seconds pendulum is very nearly a metre; it varies for points at sea level from $l_1=99.103$ cm. at the equator to $l_1=99.610$ at the poles.*

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* Further numerical data for l_1 and g will be found in J. D. EVERETT, *C. G. S. system of units*, 1891, pp. 21–22.

219. Let n be the number of small oscillations made by a pendulum of length l in the time T. Then, by (38),

$$\frac{T}{n} = \pi \sqrt{\frac{l}{g}}. \tag{40}$$

If T and one of the three quantities n, l, g in this equation be regarded as constant, the small variations of the two others can be found approximately by differentiation. For instance, if the daily number of oscillations of a pendulum of constant length be observed at two different places, we have, since T and l are constant,

$$-\frac{T}{n^2} dn = -\frac{\pi\sqrt{l}}{2} \frac{dg}{g^{\frac{3}{2}}},$$

or, dividing by (40),

$$\frac{dn}{n} = \frac{1}{2} \frac{dg}{g}. \tag{41}$$

220. Exercises.

(1) Find the number of oscillations made in a second and in a day by a pendulum 1 metre long, at a place where $g=981.0$.

(2) Find the length of the seconds pendulum at a place where $g=32.12$.

(3) To determine the value of g at a given place, the length of a pendulum was adjusted until it would make 86 400 oscillations in 24 hours. Its length was then found to be 3.3031 feet. What was the value of g ?

(4) A chandelier suspended from the ceiling of a theatre is seen to vibrate 24 times a minute. Find its distance from the ceiling.

(5) A pendulum adjusted so as to beat seconds at the equator ($g=978.1$) is carried to another latitude and is there found to make 100 oscillations more per day; find the value of g at this place.

(6) Investigate whether the approximate process of Formula (41) is sufficiently accurate for the solution of Ex. (5).

(7) If the length of a pendulum be increased by a small amount dl, show that the daily number of oscillations, n, will be decreased so that

$$\frac{dn}{n} = -\frac{1}{2} \frac{dl}{l}.$$

(8) A clock is gaining 3 minutes a day. How much should the pendulum bob be screwed up or down?

(9) A clock regulated at a place where $g=32.19$ is carried to a place where $g=32.14$. How much will it gain or lose per day if the length of its pendulum be not changed?

(10) The acceleration of gravity being inversely proportional to the distance from the earth's centre, show that the seconds pendulum will lose about 22 seconds per day if taken to a height one mile above sea level.

(11) A seconds pendulum loses 12 seconds per day, if taken to the top of a mountain. What is the height of the mountain?

(12) Show that for small oscillations the motion of a pendulum is nearly simply harmonic, and deduce from this fact the equation $t_1=\pi\sqrt{l/g}$.

221. When the oscillations of a pendulum are not so small that the arc can be substituted for its sine as was done in Art. 217, an expression for the time t_1 of one oscillation can be obtained as follows.

We have by (33), Art. 216,

$$\tfrac{1}{2}v^2-\tfrac{1}{2}v_0{}^2=gl(\cos\theta-\cos\theta_0). \qquad (33)$$

Let the time be counted from the instant when the moving point has its highest position (N in Fig. 52), so that $v_0=0$. Substituting $v=-ld\theta/dt$ and applying the formula

$$\cos\theta=1-2\sin^2\tfrac{1}{2}\theta$$

we find:

$$\tfrac{1}{2}l\left(\frac{d\theta}{dt}\right)^2=2g\left(\sin^2\frac{\theta_0}{2}-\sin^2\frac{\theta}{2}\right),$$

whence

$$dt=\tfrac{1}{2}\sqrt{\frac{l}{g}}\frac{d\theta}{\sqrt{\sin^2\dfrac{\theta_0}{2}-\sin^2\dfrac{\theta}{2}}}. \qquad (42)$$

Integrating from $\theta=0$ to $\theta=\theta_0$ and multiplying by 2 we find for the time t_1 of one oscillation:

$$t_1=\sqrt{\frac{l}{g}}\int_0^{\theta_0}\frac{d\theta}{\sqrt{\sin^2\frac{\theta_0}{2}-\sin^2\frac{\theta}{2}}}. \tag{43}$$

As θ cannot become greater than θ_0 we may put $\sin(\theta/2) = \sin(\theta_0/2)\sin\phi$, thus introducing a new variable ϕ for which the limits are 0 and $\pi/2$. Differentiating the equation of substitution, we have

$$\tfrac{1}{2}\cos\frac{\theta}{2}\,d\theta=\sin\frac{\theta_0}{2}\cos\phi\,d\phi,$$

or, as $\cos(\theta/2)=\sqrt{1-\sin^2(\theta_0/2)\sin^2\phi}$,

$$d\theta=\frac{2\sin\frac{\theta_0}{2}\cos\phi\,d\phi}{\sqrt{1-\sin^2\frac{\theta_0}{2}\sin^2\phi}}.$$

Substituting these values and putting for shortness

$$\sin\frac{\theta_0}{2}=\kappa, \tag{44}$$

we find for the time t_1 of one oscillation:

$$t_1=2\sqrt{\frac{l}{g}}\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{1-\kappa^2\sin^2\phi}}. \tag{45}$$

The integral in this expression is called the complete elliptic integral of the first species, and is usually denoted by K. Its value can be found from tables of elliptic integrals or by expanding the argument into an infinite series by the binomial theorem (since $\kappa\sin\phi$ is less than 1), and then performing the integration. We have

$$(1-\kappa^2\sin^2\phi)^{-\frac{1}{2}}=1+\tfrac{1}{2}\kappa^2\sin^2\phi+\frac{1\cdot 3}{2\cdot 4}\kappa^4\sin^4\phi+\cdots;$$

hence

$$t_1=\pi\sqrt{\frac{l}{g}}\left[1+\left(\frac{1}{2}\right)^2\kappa^2+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2\kappa^4+\cdots\right]. \tag{46}$$

If H be the height of the initial point $N(\theta=\theta_0)$ above the lowest point A of the circle, we have by (44)

$$\kappa^2 = \sin^2\frac{\theta_0}{2} = \frac{1-\cos\theta_0}{2} = \frac{H}{2l},$$

so that (46) can be written in the form

$$t_1 = \pi\sqrt{\frac{l}{g}}\left[1+\left(\frac{1}{2}\right)^2\frac{H}{2l}+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2\left(\frac{H}{2l}\right)^2+\cdots\right]. \tag{47}$$

222. Exercises.

(1) Show that $t_1 = \pi\sqrt{\frac{l}{g}}\left(1+\frac{1}{16}+\frac{9}{1024}+\frac{225}{147456}+\cdots\right)$ if the amplitude $2\theta_0$ of the oscillation is 120°.

(2) Show that as a second approximation to the time of a small oscillation we have $t_1 = \pi\sqrt{l/g}(1+\theta_0^2/16)$.

(3) Find the time of oscillation of a pendulum whose length is 1 metre at a place where $g=980.8$, to four decimal places.

(4) A pendulum hanging at rest is given an initial velocity v_1. Find to what height h_1 it will rise.

(5) Discuss the pendulum problem in the particular case when MN (Fig. 52) touches the circle at B, that is when the initial velocity is due to falling from the highest point of the circle.

223. Central Motion. The motion of a point P is called **central** if the following two conditions are fulfilled: (1) the direction of the acceleration must pass constantly through a fixed point O; (2) the magnitude of the acceleration must be a function of the distance $OP=r$ only, say

$$j=f(r). \tag{48}$$

The fixed point O is in this case usually regarded as the seat of an attractive or repulsive force producing the acceleration, and is therefore called the *centre of force.*

Harmonic motion as discussed in Arts. 172–214 is a special case of central motion, viz. the case in which the acceleration j is directly proportional to the distance from the fixed centre O, i.e. $f(r)=\mu r$.

Another very important particular case is that of *planetary motion* in which $f(r)=\mu/r^2$; this will be discussed below, Arts. 236, 239.

We proceed to establish the fundamental properties of central motion.

224. The motion is fully determined if in addition to the form of the function $f(r)$ we know the "initial conditions," say the initial distance $OP_0=r_0$ (Fig. 53) and the initial velocity v_0 of the point at the time $t=0$. As v_0 must be given both in magnitude and direction, the angle ψ_0 between r_0 and v_0 must be known.

225. It is evident, geometrically, that the motion is confined to the plane determined by O and v_0 since the acceleration always lies in this plane. This fact *that the motion is plane* depends solely on the former of the two conditions of our problem (Art. 223); that is, any motion in which the acceleration passes constantly through a fixed point is plane.

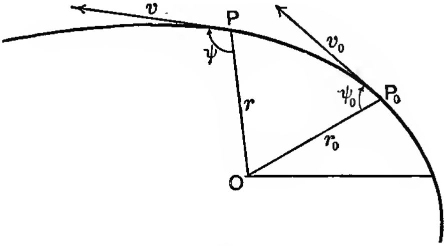

Fig. 53.

226. With O as origin, let x, y be the rectangular Cartesian co-ordinates of the moving point P, and r, θ its polar co-ordinates, at any time t. Then $\cos\theta=x/r$, $\sin\theta=y/r$ are the direction cosines of $OP=r$, and, therefore, those of the acceleration j, provided the sense of j be away from the centre, *i.e.* provided the force causing the acceleration be *repulsive.* In the case of *attraction,* the direction cosines of j are of course $-x/r$, $-y/r$.

Thus the *equations of motion* are in the case of attraction,

$$j_x \equiv \frac{d^2x}{dt^2} = -f(r)\frac{x}{r}, \quad j_y \equiv \frac{d^2y}{dt^2} = -f(r)\frac{y}{r}. \tag{49}$$

For repulsion, it would only be necessary to change the sign of $f(r)$.

227. To perform a first integration, multiply the equations (49) by y, x and subtract when the left-hand member will be found an exact derivative, while the right-hand member vanishes. Hence, integrating and denoting the constant of integration by h, we find

$$x\frac{dy}{dt} - y\frac{dx}{dt} = h, \tag{50}$$

or, introducing polar co-ordinates,

$$r^2\frac{d\theta}{dt} = h. \tag{51}$$

These equations show that *the sectorial velocity is constant,* and $= \frac{1}{2}h$ for our problem (see Art. 135 and Art. 163, Ex. (4)).

228. Let S be the sector P_0OP described by the radius vector r in the time t, so that $dS = \frac{1}{2}r^2d\theta$. Then (51) can be written in the form

$$\frac{dS}{dt} = \frac{1}{2}h, \tag{52}$$

whence integrating

$$S = \frac{1}{2}ht; \tag{53}$$

this expresses the fact that *the sector is proportional to the time in which it is described* which is of course only another way of stating the proposition of Art. 227.

The proof of the converse, viz. that if in a plane motion the areas swept out by the radius vector drawn from a fixed point be constant, the acceleration must constantly pass through that fixed point, is left to the student.

229. It is well known that Kepler had found by a careful examination of the observations available to him that *the orbits described by the planets are plane curves, and the sector described by the radius vector drawn from the sun to any planet is proportional to the time in which it is described.* This constitutes **Kepler's first law** of planetary motion.

He concluded from it that the acceleration must constantly pass through the sun.

230. To express the value of the constant of integration h in terms of the given initial conditions (Art. 224), *i.e.* by means of r_0, v_0, ψ_0, we notice that, at any time t,

$$h = r^2\frac{d\theta}{dt} = r \cdot \frac{rd\theta}{ds} \cdot \frac{ds}{dt} = r \sin\psi\, v\,; \tag{54}$$

hence for the time $t=0$, we find

$$h = v_0 r_0 \sin\psi_0. \tag{55}$$

Denoting the perpendiculars let fall from O on v_0 and v by p_0, p, we have $r_0 \sin\psi_0 = p_0$, $r\sin\psi = p$; hence also

$$h = p_0 v_0 = pv, \tag{56}$$

i.e. *the velocity at any time is inversely proportional to its distance from the centre.*

231. The equations of motions (49), if multiplied by dx/dt, dy/dt and added, give an equation in which both members are exact derivatives. On the left we find

$$\frac{d}{dt}\left[\frac{1}{2}\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}\left(\frac{dy}{dt}\right)^2\right] = \frac{d}{dt}(\tfrac{1}{2}v^2)\,;$$

on the right

$$-\frac{f(r)}{r}\left[x\frac{dx}{dt}+y\frac{dy}{dt}\right]=-\frac{f(r)}{2r}\frac{d}{dt}(x^2+y^2)=-\frac{f(r)}{2r}\frac{d(r^2)}{dt}=-f(r)\frac{dr}{dt}.$$

The equation

$$d(\tfrac{1}{2}v^2)=-f(r)\,dr \tag{57}$$

can therefore be integrated and we obtain

$$\tfrac{1}{2}v^2-\tfrac{1}{2}v_0{}^2=-\int_{r_0}^{r}f(r)dr. \tag{58}$$

232. The two methods of integrating the differential equations of motion used in Art. 227 and in Art. 231 are known, respectively, as the *principle of areas* and the *principle of energy* (or *vis viva*). The former name explains itself. The latter is due to the fact (to be more fully explained in kinetics) that if equation (58) be multiplied by the mass of the moving point, the left-hand member will represent the increase of the kinetic energy of the point during the motion.

Each of these methods of preparing the equations of motion for integration consists merely in combining the equations so as to obtain an exact derivative in the left-hand member of the resulting equation. If by this combination the right-hand member happens to vanish or to become likewise an exact derivative, an integration can at once be performed. This is the case in our problem.

233. The two equations (51) and (58) can be used to find the equation of the path. We have for *any* curvilinear motion (by (4), Art. 142)

$$v^2=\left(\frac{dr}{dt}\right)^2+r^2\left(\frac{d\theta}{dt}\right)^2=\left[\left(\frac{dr}{d\theta}\right)^2+r^2\right]\left(\frac{d\theta}{dt}\right)^2;$$

eliminating dt by means of (51) this becomes

$$v^2=\frac{h^2}{r^4}\left[\left(\frac{dr}{d\theta}\right)^2+r^2\right]=h^2\left[\left(\frac{du}{d\theta}\right)^2+u^2\right], \tag{59}$$

where $u=1/r$. Substituting for v its expression in terms of r or u, from (58), we have the differential equation of the path which is directly integrable.

Shorter methods will often suggest themselves in particular cases.

234. To solve the converse problem, viz. to find the law of acceleration when the equation of the path is given, we have only to substitute in (57) the expression of v^2 from (59). We find, with $u=1/r$,

$$f(r)=-\frac{d(\frac{1}{2}v^2)}{dr}=-\frac{d(\frac{1}{2}v^2)}{du}\frac{du}{dr}=u^2\frac{d(\frac{1}{2}v^2)}{du}$$

$$=h^2u^2\left(\frac{d^2u}{d\theta^2}+u\right). \tag{60}$$

235. Kepler in his **second law** had established the empirical fact that *the orbits of the planets are ellipses, with the sun at one of the foci.*

From this, Newton concluded that the law of acceleration must be that of the inverse square of the distance from the sun.

Equation (60) allows us to draw this conclusion. The polar equation of an ellipse referred to focus and major axis is

$$r=\frac{l}{1+e\cos\theta},$$

where $l=b^2/a=a(1-e^2)$; a, b being the semi-axes, l the semi-latus rectum, and e the eccentricity of the ellipse. Hence

$$\frac{1}{r}\equiv u=\frac{1}{l}+\frac{e}{l}\cos\theta,\ \frac{d^2u}{d\theta^2}=-\frac{e}{l}\cos\theta,$$

and (60) becomes

$$f(r)=\frac{h^2u^2}{l}=\frac{h^2}{a(1-e^2)}\cdot\frac{1}{r^2}. \tag{61}$$

236. The **third law of Kepler**, found by him likewise as an empirical fact, asserts that *the squares of the periodic times of different planets are as the cubes of the major axes of their orbits.*

From this fact Newton drew the conclusion that in the law of acceleration,

$$j \equiv f(r) = \frac{\mu}{r^2}, \tag{62}$$

the constant μ has the same value for all the planets.

Our formulæ show this as follows. Let T be the *periodic time* of any planet, *i.e.* the time of describing an ellipse whose semi-axes are a, b. Then, since the sector described in the time T is the area πab of the whole ellipse, we have by (53)

$$\pi ab = \tfrac{1}{2} h T.$$

Substituting in (61) the value of h found from this equation we have

$$f(r) = \frac{4\pi a^2 b^2}{l T^2} \cdot \frac{1}{r^2} = \frac{4\pi^2 a^3}{T^2} \cdot \frac{1}{r^2}. \tag{63}$$

Hence $$\mu = \frac{4\pi^2 a^3}{T^2} \tag{64}$$

is constant by Kepler's third law.

237. As mentioned in Art. 230, the velocity v can be expressed in terms of the perpendicular p let fall from the centre O on the tangent to the path:

$$v = \frac{h}{p} = \frac{v_0 p_0}{p}. \tag{65}$$

The acceleration j is also conveniently expressible in terms of p. We have by (57)

$$j \equiv f(r) = -\frac{d(\frac{1}{2}v^2)}{dr} = -\tfrac{1}{2} h^2 \frac{d}{dr}\left(\frac{1}{p^2}\right) = \frac{h^2}{p^3}\frac{dp}{dr}. \tag{66}$$

238. Finally, another expression for the acceleration is sometimes found convenient. In *any* motion, the component of the acceleration along the radius vector is (see Art. 161)

$$j = \frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2.$$

As in our problem the total acceleration is along the radius vector, in the sense towards the origin, we have

$$j \equiv f(r) = -\frac{d^2r}{dt^2} + r\left(\frac{d\theta}{dt}\right)^2,$$

or since, by (51), $d\theta/dt = h/r^2$,

$$j \equiv f(r) = -\frac{d^2r}{dt^2} + \frac{h^2}{r^3}. \tag{67}$$

The first term is what the acceleration would be if the motion were rectilinear along the radius vector; the second term represents what is due to the curvature of the path.

239. Planetary Motion, in its simplest form, is (see Art. 223) that particular case of central motion in which the acceleration is inversely proportional to the distance from the centre O, so that

$$j \equiv f(r) = \frac{\mu}{r^2},$$

where μ is a constant, viz. the acceleration at the distance $r = 1$ from O.

The equations of motion (49) are in this case, with O as origin,

$$\frac{d^2x}{dt^2} = -\mu\frac{x}{r^3}, \quad \frac{d^2y}{dt^2} = -\mu\frac{y}{r^3}. \tag{68}$$

Combining these by the principle of energy (Arts. 231, 232), we find

$$\frac{d(\frac{1}{2}v^2)}{dt} = -\frac{\mu}{r^3}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right) = -\frac{\mu}{r^3}\frac{d}{dt}\left(\frac{x^2+y^2}{2}\right)$$

$$= -\frac{\mu}{r^2}\frac{dr}{dt} = \mu\frac{d\left(\frac{1}{r}\right)}{dt};$$

hence integrating

$$\tfrac{1}{2}v^2 - \tfrac{1}{2}v_0^2 = \frac{\mu}{r} - \frac{\mu}{r_0}. \tag{69}$$

240. To find the equation of the path, or *orbit*, we write the equations (68) in the form

$$\frac{d^2x}{dt^2}=-\frac{\mu}{r^2}\cos\theta,\quad \frac{d^2y}{dt^2}=-\frac{\mu}{r^2}\sin\theta,$$

and eliminate r^2 by means of (51):

$$\frac{d^2x}{dt^2}=-\frac{\mu}{h}\cos\theta\frac{d\theta}{dt},\quad \frac{d^2y}{dt^2}=-\frac{\mu}{h}\sin\theta\frac{d\theta}{dt}.$$

These equations can be integrated separately:

$$\frac{dx}{dt}-v_1=-\frac{\mu}{h}\sin\theta,\quad \frac{dy}{dt}-v_2=\frac{\mu}{h}(\cos\theta-1) \tag{70}$$

where v_1, v_2 are the components of the initial velocity.

Multiplying by y, x and subtracting, we find, owing to (50),

$$\left(\frac{\mu}{h}-v_2\right)x+v_1y+h=\frac{\mu}{h}(x\cos\theta+y\sin\theta)=\frac{\mu}{h}\sqrt{x^2+y^2}. \tag{71}$$

241. The geometrical meaning of this equation is that the radius vector $r=\sqrt{x^2+y^2}$ drawn from the fixed point O to the moving point P is proportional to the distance of P from the fixed straight line

$$\left(\frac{\mu}{h}-v_2\right)x+v_1y+h=0. \tag{72}$$

It represents, therefore, a conic section having O for a focus and the line (72) for the corresponding directrix.

The character of the conic depends on the absolute value of the ratio of the radius vector to the distance from the directrix; according as this ratio

$$\frac{h}{\mu}\sqrt{\left(\frac{\mu}{h}-v_2\right)^2+v_1^2}\lesseqgtr 1,$$

the conic will be an ellipse, a parabola, or a hyperbola. The criterion can be simplified. Multiplying by μ/h and squaring, we have

$$-\frac{2\mu v_2}{h}+v_2^2+v_1^2\lesseqgtr 0,$$

or since $v_1^2+v_2^2=v_0^2$ and $h=r_0v_0\sin\psi_0=r_0v_2$:

$$v_0^2 \begin{smallmatrix}<\\=\\>\end{smallmatrix} \frac{2\mu}{r_0}. \tag{73}$$

242. Introducing polar co-ordinates in (71), the equation of the orbit assumes the form

$$\frac{1}{r}=\frac{\mu}{h^2}+\left(\frac{v_2}{h}-\frac{\mu}{h^2}\right)\cos\theta-\frac{v_1}{h}\sin\theta,$$

or putting $(hv_2-\mu)/h^2=C\cos\alpha$, $v_1/h=C\sin\alpha$,

$$\frac{1}{r}=\frac{\mu}{h^2}+C\cos(\theta+\alpha). \tag{74}$$

This equation might have been obtained directly by integrating (60), which in our case, with $f(r)=\mu/r^2$, reduces to

$$\frac{d^2u}{d\theta^2}+u=\frac{\mu}{h^2};$$

the general integral of this differential equation is of the form (74), C and α being the constants of integration.

Equation (74) represents a conic section referred to the focus as origin and a line making an angle α with the focal axis as polar axis.

243. Exercises.

(1) If $2k$ be the chord intercepted by the osculating circle on the radius vector drawn from the fixed centre, show that $v^2=k\cdot f(r)$.

(2) A point moves in a circle; if the acceleration be constant in direction, what is its magnitude?

(3) A point moves in a circle; if the acceleration be constantly directed towards the centre, what is its magnitude?

(4) A point is subject to a central acceleration proportional to the distance from the centre and directed away from the centre; find the equation of the path.

(5) A point P is subject to two accelerations, $-\mu^2\cdot O_1P$ directed towards the fixed point O_1, and $-\mu^2\cdot O_2P$ directed away from the fixed point O_2. Show that its path is parabolic.

(6) A point P describes an ellipse owing to a central acceleration $f(r) = \mu/r^2$ directed toward the focus S. Its initial velocity v_0 makes an angle ψ_0 with the initial radius vector r_0. Determine the semi-axes a, b of the ellipse in magnitude and position.

244. The student will find numerous examples for further practice in the kinematics of a particle in the following works: P. G. TAIT and W. J. STEELE, *A treatise on dynamics of a particle*, 6th ed., London, Macmillan, 1889; W. H. BESANT, *A treatise on dynamics*, London, Bell, 1885; B. WILLIAMSON and F. A. TARLETON, *An elementary treatise on dynamics*, 2d ed., London, Longmans, 1889; W. WALTON, *Collection of problems in illustration of the principles of theoretical mechanics*, 3d ed., Cambridge, Deighton, 1876.

5. VELOCITIES IN THE RIGID BODY.

245. A rigid body is said to have plane motion when all its points move in parallel planes. Its motion is then fully determined by the motion of any plane section of the body in its plane.

It has been shown in Arts. 18–24 that the continuous motion of an invariable plane figure in its plane consists in a series of infinitesimal rotations about the successive instantaneous centres, *i.e.* about the points of the space centrode.

If at any instant the centre of rotation and the angular velocity ω about it be known, we can find *the velocity of any point of the plane figure.*

To show this let us first take the instantaneous centre as origin. Then the component velocities v_x, v_y of any point P whose co-ordinates are x, y, or r, θ, are found (Art. 141) by differentiating the expressions

$$x = r\cos\theta, \quad y = r\sin\theta$$

with respect to t. Considering that $d\theta/dt$ is the angular velocity ω about the instantaneous centre, we find

$$\begin{aligned} v_x &= \frac{dx}{dt} = -r\sin\theta\frac{d\theta}{dt} = -\omega y, \\ v_y &= \frac{dy}{dt} = \quad r\cos\theta\frac{d\theta}{dt} = \quad \omega x. \end{aligned} \tag{1}$$

246. Next, taking an arbitrary origin O (Fig. 54), let x, y be the co-ordinates of P and x'_0, y'_0 those of any other point O' of the moving figure with respect to fixed rectangular axes through O; and let ξ, η be the co-ordinates of P with reference to rectangular axes through O' fixed in the figure but moving with it. Then, if θ be the angle between the axes Ox and $O'\xi$, we have

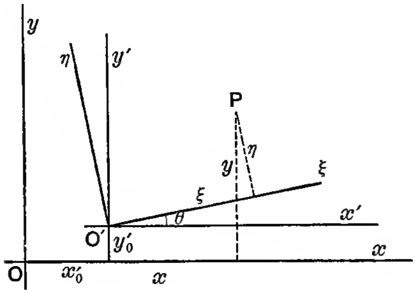

Fig. 54.

$$x = x' + \xi\cos\theta - \eta\sin\theta, \quad y = y' + \xi\sin\theta + \eta\cos\theta.$$

Differentiating we find for the component velocities of P parallel to the fixed axes Ox, Oy:

$$v_x = \frac{dx'}{dt} - (\xi\sin\theta + \eta\cos\theta)\frac{d\theta}{dt}, \quad v_y = \frac{dy'}{dt} + (\xi\cos\theta - \eta\sin\theta)\frac{d\theta}{dt}. \tag{2}$$

Now, $d\theta/dt$ is the angular velocity ω about the point O' while dx'/dt, dy'/dt are the velocities of O' parallel to the fixed axes, say $v_{x'}$, $v_{y'}$. Considering moreover that $\xi\sin\theta + \eta\cos\theta = y - y'$, $\xi\cos\theta - \eta\sin\theta = x - x'$, we have

$$v_x = v_{x'} - (y - y')\omega, \quad v = v_{y'} + (x - x')\omega. \tag{3}$$

The velocity of P *consists*, therefore, *of two parts, a velocity of translation equal to that of* O′ *and a velocity of rotation equal to that of* P *about* O′.

247. The instantaneous centre being the point whose velocity is zero at the given instant, we find its co-ordinates x_0, y_0 from the equations

$$0 = v_{x'} - (y_0 - y')\omega, \quad 0 = v_{y'} + (x_0 - x')\omega,$$

whence
$$x_0 = x' - \frac{v_{y'}}{\omega}, \quad y_0 = y' + \frac{v_{x'}}{\omega}. \tag{4}$$

By eliminating t between these equations, the equation of the space centrode can be found.

The co-ordinates ξ_0, η_0 of the instantaneous centre referred to the moving axes are found in a similar way from the equations (2):

$$\xi_0 = \frac{1}{\omega}(v_{x'} \sin\theta - v_{y'} \cos\theta), \quad \eta_0 = \frac{1}{\omega}(v_{x'} \cos\theta + v_{y'} \sin\theta), \tag{5}$$

from which the body centrode can be found by eliminating t.

248. In Arts. 245 and 246 expressions were found for the component velocities v_x, v_y parallel to the fixed axes Ox, Oy. To find the component velocities v_ξ, v_η parallel to the moving axes $O\xi$, $O\eta$, let x, y be the co-ordinates of any point P with respect to the fixed axes (Fig. 55), ξ, η those with respect to the moving axes, and let θ be the angle $xO\xi$. The velocity of P parallel to the axes $O\xi$, $O\eta$ consists of two parts, that arising from the motion of P relative to $\xi O\eta$ whose components are of course $d\xi/dt$, $d\eta/dt$, and that due to the rotation of the moving axes. The components of the latter velocity are, by (1), Art. 245, $-\omega\eta$, $\omega\xi$. Hence

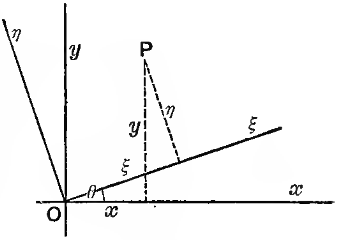

Fig. 55.

$$v_\xi = \frac{d\xi}{dt} - \omega\eta, \quad v_\eta = \frac{d\eta}{dt} + \omega\xi. \tag{6}$$

249. Exercises.

(1) Two points A, A' of a plane figure move on two fixed circles described with radii a, a' about O, O'; show that the angular velocities ω, ω' of OA, $O'A'$ about O, O' are inversely proportional to OM, $O'M$, M being the point of intersection of OO' with AA'.

(2) Given the magnitudes v, v' of the velocities of two points A, A' of an invariable plane figure and the angle (v, v') formed by their directions; find the instantaneous centre C and the angular velocity ω about C.

(3) Show that in the "elliptic motion" of a plane figure (Arts. 25–27) the velocity of any point (x', y') is

$$v = [a^2 + x'^2 + y'^2 - 2a(x' \cos 2\phi + y' \sin 2\phi)]^{\frac{1}{2}} \frac{d\phi}{dt}.$$

(4) In the same motion find the velocities of B and O' (Fig. 6, Art. 26) when A moves uniformly along the axis of x.

250. The continuous motion of a rigid body is called a **translation** when the velocities of all its points are equal and parallel at every moment (Art. 9). All points describe therefore equal and similar curves, and every line of the body remains parallel to itself. The velocity $v = ds/dt$ of any point is called the velocity of translation of the body.

251. A rigid body can be imagined to be subjected to several velocities of translation simultaneously; the resulting motion is a translation whose velocity is found by geometrically adding the component velocities.

Conversely, the velocity of translation of a rigid body can be resolved into components in given directions.

252. The continuous motion of a rigid body is called **rotation** when two points of the body are fixed; the line joining these points is the axis of rotation. All points excepting those on the axis describe arcs of circles whose centres lie on the axis.

The velocity of any point P of the body at the distance $OP = r$ from the axis is $v = \omega r = r\, d\theta/dt$, if $\omega = d\theta/dt$ is the angular velocity of the rigid body. The velocities of the different points of the body at any given moment are therefore directly proportional to their distances from the axis, and the velocities of all points at this moment are known if the instantaneous angular velocity ω is given. It is frequently convenient to imagine this angular velocity represented by its *rotor*, *i.e.* by a length ω laid off in the proper sense on the axis of rotation (see Arts. 68, 69).

253. The body may have several simultaneous rotations. Imagine, for instance, a top spinning about its axis placed on a table or disc which is made to rotate about an axis. The resulting motion can be found by compounding the rotors in the same way in which the rotors representing infinitesimal rotary displacements are compounded (Arts. 62, 66, 67); indeed, the rotor $\omega = d\theta/dt$ of an angular velocity is merely the rotor $d\theta$ divided by dt, and therefore identical with the rotor of the angular displacement $d\theta$.

254. As we are at present concerned with plane motion, we require only the rule for the composition of angular velocities about *parallel* axes.

Dividing the equations (1‴) and (2‴) of Art. 66 by dt, and putting $d\theta/dt = \omega$, $d\theta_1/dt = \omega_1$, $d\theta_2/dt = \omega_2$, we obtain:

$$\omega = \omega_1 + \omega_2, \quad \frac{L_1L}{\omega_2} = \frac{LL_2}{\omega_1} = \frac{L_1L_2}{\omega}. \tag{7}$$

Thus, *the resultant of two angular velocities* ω_1, ω_2 *about parallel axes* l_1, l_2 *is an angular velocity* ω *equal to their algebraic sum,* $\omega = \omega_1 + \omega_2$, *about a parallel axis* l *that divides the distance between* l_1, l_2 *in the inverse ratio of* ω_1 *and* ω_2.

Conversely, an angular velocity ω about an axis l can always be replaced by two angular velocities ω_1, ω_2 whose sum is equal to ω and whose axes l_1, l_2 are parallel to l and so selected that l divides the distance between l_1, l_2 inversely as ω_1 is to ω_2.

255. It may be well to prove this important proposition independently. Any point P (Fig. 56) in a plane at right angles to the axes receives from ω_1 a linear velocity $\omega_1 r_1$ perpendicular to L_1P, and from ω_2 a linear velocity $\omega_2 r_2$ perpendicular to L_2P, if $L_1P = r_1$, $L_2P = r_2$. These linear velocities fall into the same straight line only for points situated on the line L_1L_2. A point L whose linear velocity is zero, must therefore lie on L_1L_2 so that $L_1L + LL_2 = L_1L_2$; moreover, it must satisfy the condition $L_1L \cdot \omega_1 = LL_2 \cdot \omega_2$. This gives the above equations (7).

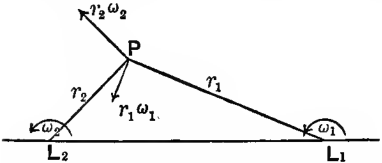

Fig. 56.

256. The resulting axis lies between L_1 and L_2 when the components ω_1, ω_2 have the same sense; when they are of opposite sense, it lies without, on the side of the greater one of these components.

If ω_1 and ω_2 are equal and opposite, say $\omega_1 = \omega$, $\omega_2 = -\omega$, the resulting axis lies at infinity (Art. 67). Two such equal and opposite angular velocities about parallel axes are said to form a **rotor-couple**; its effect on the rigid body is that of a velocity of translation $v = L_1L_2 \cdot d\theta/dt = p \cdot \omega$ at right angles to the plane of the axes. The distance of the rotor, $L_1L_2 = p$, is called the *arm* of the couple, and the product $p\omega = v$ its *moment.*

257. A velocity of translation v can therefore always be replaced by a rotor-couple $p\omega = v$, whose axes have the distance p and lie in a plane at right angles to v.

Again, an angular velocity ω about an axis l can be replaced by an equal angular velocity ω about a parallel axis l' at the distance p from l, in combination with a velocity of translation $v = \omega p$ at right angles to the plane determined by l and l'.

It easily follows from these propositions that *the resultant of any number of velocities of translation,* v, v′, …, *parallel to the same plane, and any number of angular velocities* ω, ω', …, *about* *axes perpendicular to this plane is always a single angular velocity about an axis perpendicular to the plane or a single velocity of translation parallel to the plane.*

6. APPLICATIONS.

258. Kinematics of Machinery. A large majority of the cases of motion that are of importance in mechanical engineering can be reduced to plane motion.

At first glance the application of theoretical kinematics to machines might seem to lead to rather complicated problems owing to the fact that a machine is never formed by a single rigid body, but always consists of an assemblage of several bodies some of which may even be not rigid (belting, springs, water, steam). The problem is, however, very much simplified by a characteristic of all machines, properly so called, that was first pointed out and insisted upon by recent writers on applied kinematics, in particular by Reuleaux.* This characteristic is the **constrainment** of the motions of the parts of a machine.

Thus Professor Kennedy defines a machine as "a combination of resistant bodies whose relative motions are completely constrained, and by means of which the natural energies at our disposal may be transformed into any special form of work."

With the latter clause of this definition we are not at present concerned; it will be considered in kinetics. To explain the former in detail would lead us too far into the domain of applied mechanics. A brief indication of the fundamental ideas must be sufficient.

259. By considering machines of various types it appears that the bodies, or *elements*, composing a machine always occur in **pairs**. Thus a single rigid bar will form a lever only when taken in connection with a support, or fulcrum; a shaft to be used in a machine must rest in bearings; a screw must turn in a nut. To take a more complex illustration, consider the mechanism formed by the crank and connecting rod of a steam-engine (Fig. 57). It may be regarded as composed of three pairs, two so-called turning pairs at O and A, and a sliding pair at B; and these three pairs are connected by three rigid bars, called **links**, OA, AB, OB, the last of which is fixed.

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* F. REULEAUX, *Theoretische Kinematik*, Berlin, 1875; translated into English and edited by ALEX. B. W. KENNEDY under the title *Kinematics of machinery*, London, Macmillan, 1876. Compare also R. WILLIS, *Principles of mechanism*, London, Longmans, 2d ed. 1870 (1st ed. 1841); F. GRASHOF, *Theoretische Maschinenlehre*, Vol. II., Leipzig, Voss, 1883; L. BURMESTER, *Lehrbuch der Kinematik*, Leipzig, Felix, 1888; ALEX. B. W. KENNEDY, *Mechanics of machinery*, London, Macmillan, 1886; J. H. COTTERILL, *Applied mechanics*, London, Macmillan, 1884.

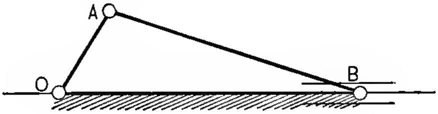

Fig. 57.

260. A **sliding pair** is formed by two bodies so connected that one is constrained to have a motion of translation relatively to the other. A pin moving in a groove or slot, a sleeve sliding along a shaft, are familiar examples.

A **turning pair** constrains one body to rotate about a fixed axis in another, as in the case of a shaft turning in its bearings.

A **twisting pair** makes one body have a screw motion about an axis fixed in the other.

These three pairs are the only so-called **lower pairs.** They are characterized as such by the fact that their elements have surface contact, and that, if either element be fixed, every point of the other is constrained to move in a definite line. In other words, the constraint effected by lower pairing is such as to leave but one degree of freedom (see Art. 37) to either element if the other be fixed.

261. All other pairs are called **higher pairs.** The contact in such pairs is usually line contact, and the two bodies have more than one degree of freedom relatively to each other, usually two degrees, so that if one element be fixed, any point of the other is constrained to a surface.

Higher pairs are of far less frequent occurrence in ordinary machines than lower pairs. The only very common example of higher pairing is found in toothed wheel gearing.

262. For the purposes of kinematics a machine may be regarded as consisting of a number of bodies (*links*) connected by pairs in such a way that when one of the links is fixed all other links are constrained in their motion. In most cases this constraint is such as to leave but one degree of freedom to every link.

A system of links of this kind forming, so to speak, a skeleton of the machine is called a **kinematic chain** (Reuleaux). When one link of such a chain is fixed, the chain becomes a **mechanism**. As a typical example we may take the "slider crank" in Fig. 57.

If the pairs are all turning pairs with parallel axes, the chain is called a **linkage** (Sylvester). A typical example is the four bar linkage in Fig. 58. A linkage with one link fixed has been called a **linkwork** (Sylvester). The four bar linkwork in Fig. 58 is also called a "lever crank" (Kennedy).

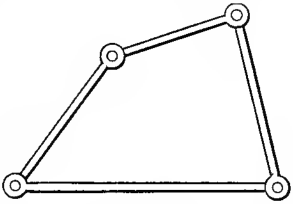
Fig. 58.

263. The Four Bar Linkage 1 2 3 4 (Fig. 59). Whatever may be its motion, each link considered separately moves as an invariable plane figure and has therefore at any moment an instantaneous centre C and an angular velocity ω about this centre.

The centre C_{12} of 1 2 and the centre C_{23} of 2 3 must always lie on a line passing through 2 since the velocity of 2 is perpendicular to both $C_{12}2$ and $C_{23}2$.

Similarly, 3 must lie on the line joining the centres C_{23} and C_{34}; and so on.

The quadrilateral 1 2 3 4 is therefore, and always remains, inscribed in the quadrangle $C_{12}C_{23}C_{34}C_{41}$. This can be shown to hold even for the *complete* quadrilateral and quadrangle. The complete quadrilateral, or four-side, 1 2 3 4 has six vertices, viz. the six intersections 1, 2, 3, 4, 5, 6 of its four sides; the complete quadrangle, or four-point, $C_{12}C_{23}C_{34}C_{41}$ has six sides, viz. the six lines $C_{41}C_{12}$, $C_{12}C_{23}$, $C_{23}C_{34}$, $C_{34}C_{41}$, $C_{12}C_{34}$, $C_{23}C_{41}$ joining its four vertices; and these six sides of the quadrangle pass through the six vertices of the quadrilateral, respectively.

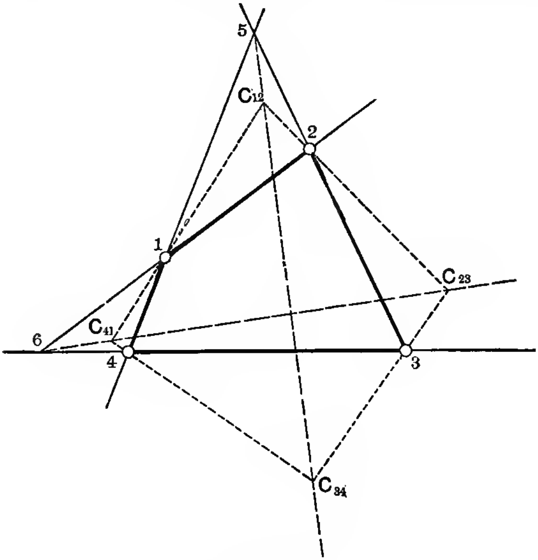

Fig. 59.

It remains to prove that $C_{12}C_{34}$ passes through 5 and that $C_{23}C_{41}$ passes through 6.

Now the velocity of 2 can be expressed by $\omega_1 \cdot C_{12}2$ and also by $\omega_2 \cdot C_{23}2$; hence $C_{12}2/C_{23}2 = \omega_2/\omega_1$; similarly $C_{23}3/C_{34}3 = \omega_3/\omega_2$. We have therefore, by the proposition of Menelaus,* for the intersection of 2 3 with $C_{12}C_{34}$:

$$\frac{5\,C_{12}}{5\,C_{34}} = \frac{\omega_3}{\omega_1}.$$

The same value is obtained by determining the intersection of 1 4 with $C_{12}C_{34}$; the two intersections must therefore coincide.

The proof for the point 6 is analogous.

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* If the sides of a triangle ABC be cut by any transversal, in the points A', B', C', then $\frac{BA'}{A'C} \cdot \frac{CB'}{B'A} \cdot \frac{AC'}{C'B} = -1$. See J. Casey, *Sequel to Euclid*, London, Longmans, 1882, p. 69.

A corresponding proposition holds of course for four bar linkages with crossed bars 1 2 4 3, or 1 3 2 4.

264. Lever-crank. The linkage considered in the preceding article becomes a mechanism, or linkwork, as soon as one of its four links is fixed. It occurs in machines under a variety of forms some of which are referred to below.

Let the link 3 4 be fixed; then the centre C_{34} (Fig. 59) disappears; C_{41} falls into 4, C_{23} into 3, and C_{12} becomes the intersection 5 of 4 1 and 3 2. If 1 2 were fixed instead of 3 4, 3 4 would have its centre at 5.

Similarly, if either 4 1 or 2 3 be fixed, the centre of the other is 6.

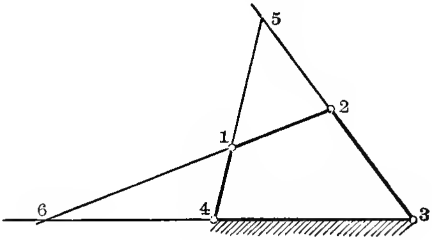

Fig. 60.

Hence whichever of the four links be fixed, the centres of all the links lie at some of the six vertices of the complete quadrilateral 1 2 3 4.

If 3 4 be the fixed link (Fig. 60), the ratio of the angular velocities ω_1 of 4 1 and ω_2 of 3 2 can be found. For if ω denote the angular velocity of 1 2 about 5, we have

$$4\,1\cdot\omega_1=5\,1\cdot\omega,\quad 3\,2\cdot\omega_2=5\,2\cdot\omega;$$

hence
$$\frac{\omega_2}{\omega_1}=\frac{4\,1}{3\,2}\cdot\frac{5\,2}{5\,1}=\frac{5\,2}{3\,2}\Big/\frac{5\,1}{4\,1};$$

or, by the proposition of Menelaus:

$$\frac{\omega_2}{\omega_1}=\frac{4\,6}{3\,6}.$$

265. Parallelogram: 4 1 = 3 2 = a, 4 3 = 1 2 = b (Fig. 61). The link 1 2 has evidently a motion of translation, its instantaneous centre lying at the intersection of the parallel lines 4 1, 3 2.

The space centrode is the line at infinity; the body centrode may be regarded as a circle of infinite radius described about the midpoint of 3 4 as centre.

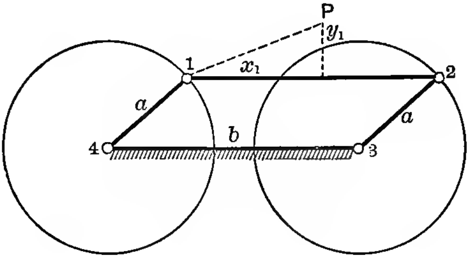

Fig. 61.

To find the equation of the path of any point P rigidly connected with 1 2, let x, y be the rectangular co-ordinates, with respect to 4 as origin and 4 3 as axis of x, and x_1, y_1 its co-ordinates for parallel axes through 1; then, putting $\measuredangle$ 3 4 1 = θ, we have

$$x = a\cos\theta + x_1, \quad y = a\sin\theta + y_1;$$

hence, eliminating θ,

$$(x - x_1)^2 + (y - y_1)^2 = a^2,$$

which represents a circle of radius a whose centre has the fixed co-ordinates x_1, y_1.

For the velocity of P we have $dx/dt = -a\omega\sin\theta$, $dy/dt = a\omega\cos\theta$; hence $v = a\omega$, as is otherwise apparent.

266. If in the parallelogram 1 2 3 4 the point 4 alone be fixed, we have a linkage called the **pantograph.**

It can serve to trace a curve similar to a given curve. Indeed, any line through 4 (Fig. 62) cuts the opposite links

1 2, 2 3 (produced if necessary) in points A, A' whose paths are homothetic (similar and similarly situated) curves. For the points 4, A, A' remain always in line and the ratio $4A/4A'$ remains constant. Hence if a pencil be attached to A' and A be made to trace a given curve, A' will trace a similar curve.

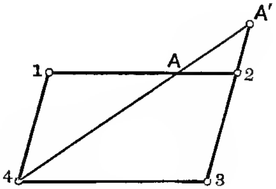

Fig. 62.

Instead of fixing 4, the point A' might be fixed; then 4 and A will describe similar curves. This property is utilised in Watt's parallel motion (see Art. 271).

The parallelogram linkage furnishes also a simple instrument for describing ellipses. Let the sides of the parallelogram be $23 = 41 = a$, $12 = 34 = b$; and let a point A' on 2 3 produced, at the distance b from 2, be fixed (Fig. 63). Then, if 1 be made to describe a straight line, passing through A', 4 will describe an ellipse. For, taking A' as origin and $A'1$ as axis of x, we

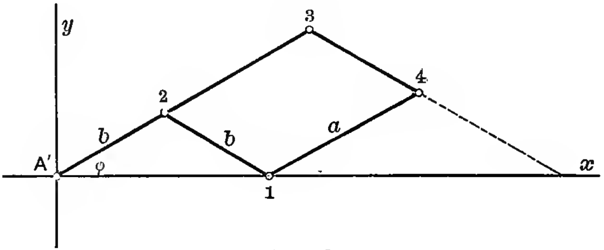

Fig. 63.

have for the co-ordinates of 4: $x = (a + 2b) \cos\phi$, $y = a \sin\phi$, whence

$$\frac{x^2}{(a+2b)^2} + \frac{y^2}{a^2} = 1.$$

267. In the parallelogram 1 2 3 4, let the link 1 2 be turned so as to coincide in direction with 4 3, and then give the links 4 1 and 3 2 rotations of opposite sense. We thus obtain a linkage with equal, but intersecting, opposite sides, which we may call **anti-parallelogram** (Fig. 64). If 3 4 be fixed, the instantaneous centre of 1 2 is the intersection 5 of 4 1 and 2 3.

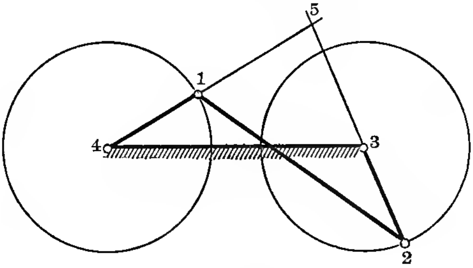

Fig. 64.

To obtain the centrodes in this case, notice that as the triangles 1 5 2 and 5 3 4 are equal, the triangle 5 4 2 is isosceles; hence 5 1 = 5 3, and 4 5 − 3 5 = 4 1 = a. The difference of the radii vectores of 5 drawn from 4 and 3 being thus constant, it follows that the space centrode is a hyperbola whose foci are 4, 3, and whose real axis $= a$. As 4 3 = 1 2 = b, the equation of this hyperbola is

$$\frac{x^2}{\left(\frac{a}{2}\right)^2} - \frac{y^2}{\frac{b^2 - a^2}{4}} = 1,$$

for 4 3 as axis of x and the midpoint of 4 3 as origin.

It is easy to see that the space centrode becomes an ellipse when $b < a$.

As the triangles 1 5 2 and 3 5 4 are equal the body centrode is an equal hyperbola or ellipse. The two centrodes lie symmetrically with respect to their common tangent at 5.

For a given anti-parallelogram the centrodes are hyperbolas when one of the larger links is fixed; they are ellipses when one of the shorter links is fixed.

268. If in the anti-parallelogram only one point, say 4, be fixed, it can be used as an **inversor**, *i.e.* as an instrument for describing the inverse of a given curve.

Let $r=OP$ be the radius vector drawn from an arbitrary fixed origin, or pole, O to a given curve; on OP lay off a length $OP' \equiv r' = \kappa^2/r$, where κ is a constant; then P' is said to describe the *inverse* of the given curve.

The theory of inversors is based on the following geometrical proposition: if three lines $CA=a$, $CA'=a$, $CO=b$ (Fig. 65) turn about C so that O, A, A' are always in line, the product $OA \cdot OA'$ remains constant, viz. $OA \cdot OA' = b^2-a^2$. For if the circle of radius a described about C intersect the line OC in B and B' we have $OA \cdot OA' = OB \cdot OB' = (b-a)(b+a)$.

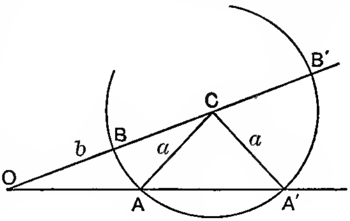

Fig. 65.

This proposition shows that in the anti-parallelogram 1 2 3 4 (Fig. 66), with the vertex 4 fixed, the line joining the vertices 4 and 2 intersects the circle described about 3 with radius 3 2 in a point 2′ such that 2 and 2′ describe inverse curves with respect to 4 as pole. For we have $4\,2' \cdot 4\,2 = 4\,3^2 - 2\,3^2 = b^2 - a^2$.

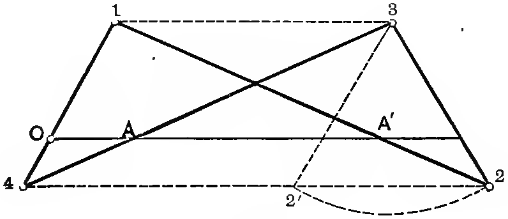

Fig. 66.

Moreover, any parallel to 4 2 will intersect the links 4 1, 4 3, 2 1 in points O, A, A' dividing the three lines in the same ratio; hence if O be fixed, A and A' will describe inverse curves for O as pole. This is the principle on which **Hart's inversor** is based.

269. Peaucellier's cell is another inversor (Fig. 67). It consists of the linked rhombus $ABA'B'$ whose side we denote by a, and the two equal links OB, OB' of length b. If O be fixed, A and A' evidently describe inverse curves for O as pole.

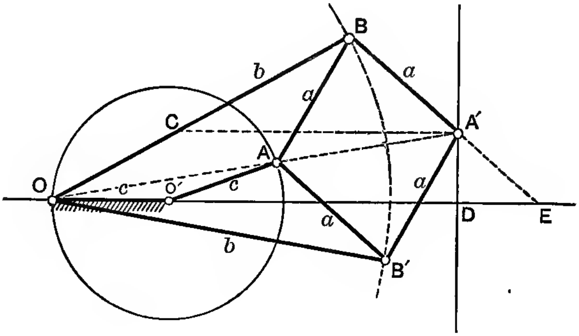

Fig. 67.

The practical application of inversors is based on the property that they enable us to transform circular motion into rectilinear motion (see Art. 271).

The inverse of a circle $r = 2c\cos\theta$ passing through the pole is a straight line; for we have for the radius vector r' of the inverse curve $r' = \kappa^2/r = \kappa^2/2c\cos\theta$; hence $r'\cos\theta = \kappa^2/2c$ which is the equation of a straight line at right angles to the polar axis, at the distance $\kappa^2/2c$ from the pole.

If therefore the point A of an inversor be made to describe an arc of a circle passing through O, the point A' will describe a segment of a straight line. The vertex A (Fig. 67) can be compelled to describe a circle by inserting the additional link $O'A$ turning about the fixed point O'. If O' be selected so as to make $O'O = O'A$, say $= c$, the circle described by A will pass through O; and the motion of A' will be confined to the straight line $A'D$ perpendicular to OO', at the distance $OD = (b^2 - a^2)/2c$ from O.

The linkage has thus become a linkwork, OO' being the fixed link.

270. To determine the linear velocity v of A' along DA' when the angular velocity ω of the link OB is given, we notice that the instantaneous centre C of the link BA' lies at the intersection of OB with the line drawn through A' parallel to OO'. Let ω' be the angular velocity of BA' about C. Then $v=\omega' \cdot CA'$; also since the point B describes a circle about O, $\omega b=\omega' \cdot CB$; hence

$$v=\omega \cdot \frac{CA'}{CB} b.$$

If BA' intersect OO' in E, we have from similar triangles $CA' : CB = OE : OB$; hence

$$v=\omega \cdot OE.$$

The variable length OE depends on the angles $EOB=\theta$ and $BEO=\phi$ which are connected by the relation (Art. 269)

$$a \cos \phi + b \cos \theta = OD = \frac{b^2-a^2}{2c}.$$

The figure gives $OE=b \cos \theta + b \sin \theta \cot \phi$; hence, finally,

$$v=\omega b \sin \theta (\cot \theta + \cot \phi).$$

271. In the steam engine and other machines mechanisms are required for transforming the alternating rectilinear motion of the piston into the reciprocating circular motion of a crank, eccentric, or beam; a mechanism of this kind is called, rather inappropriately, a **parallel motion.** The problem of effecting this transformation has been solved in various ways. Peaucellier's inversor (1864) was the first *accurate* solution. Generally, an approximate solution is sufficient for practical purposes. The most common of such approximations is **Watt's parallel motion.** This mechanism is a combination of a linked parallelogram with a four bar linkwork with crossed links.

To fix the ideas, let 4 1 (Fig. 68) be the horizontal middle position of the beam of a beam engine; 4 is fixed and 1 describes an arc of a circle of radius 4 1 $= a$. We might place a counter-beam 3 2 of equal length turning about the fixed end 3 so as to be in its middle position parallel to 4 1 and so as to make the connecting link 1 2 nearly vertical. The middle point of 1 2 would then describe a looped curve whose central portion does not differ very much from a straight line; connecting this middle point with the piston rod, the problem would be solved.

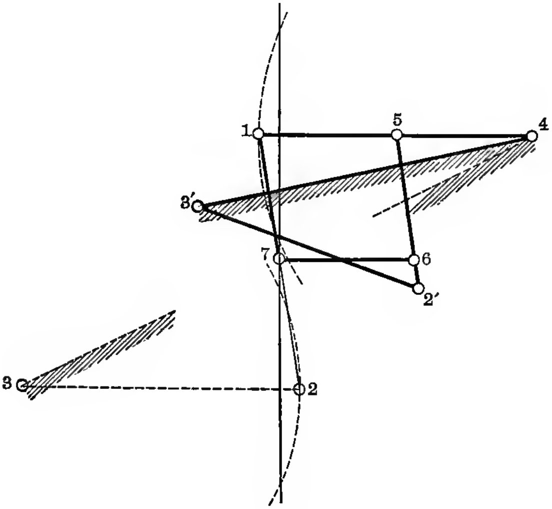

Fig. 68.

But the introduction of the large counter-beam 3 2 in the position indicated above would be very inconvenient. To reduce the size of the mechanism the counter-beam 3 2 is placed nearer to 4 1, into the position 3′ 2′, and the parallelogram 1 5 6 7 is introduced, the piston rod being attached at 7. Owing to the property of the linked parallelogram (Art. 266), the point 7 has a motion similar to that of the point of intersection of 4 7 with 5 6; it describes therefore approximately a straight line. The point of intersection of 47 with 56 can be used to connect with the pump rods of the engine.

7. ACCELERATIONS IN THE RIGID BODY.

272. To find the accelerations of the various points of a rigid body we must compare the velocities of these points during two consecutive elements of time; the change of the velocity divided by dt gives the acceleration.

In the case of **translation** (Art. 250) the accelerations of all points of the body are evidently equal so that the acceleration of any point may be called the acceleration of the body.

273. In the case of **rotation about a fixed axis** l, any point P of the body at the distance r from the axis describes during the element of time dt a space element $ds = rd\theta = \omega r dt$ proportional to this distance r, where $\omega = d\theta/dt$ is the angular velocity of the body about the axis l. The linear velocity of P is $v = \omega r$. The space element ds' described during the next element of time is an infinitesimal arc of the same circle of radius r, *i.e.*

$$ds' = rd\theta' = r(\omega + d\omega)dt.$$

Drawing from any point O (Fig. 69) the vectors $OV = ds/dt$, $OV' = ds'/dt$, and resolving the elementary acceleration VV' parallel to the tangent and normal of the path into $TV' = dv = rd\omega$ and $VT = vd\theta = r\omega d\theta = r\omega^2 dt$, we find the tangential and normal components of the acceleration of P by dividing these elements by dt. Hence denoting the angular acceleration $d\omega/dt$ by α, we have

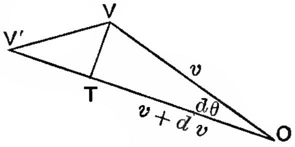

Fig. 69.

$$j_t = \alpha r, \quad j_n = \omega^2 r. \tag{1}$$

The total acceleration of P,

$$j = \sqrt{j_t^2 + j_n^2} = r\sqrt{\alpha^2 + \omega^4}, \tag{2}$$

is therefore proportional to the distance r of this point from the axis, so that the accelerations of all points can be found as soon as that of any one point is known.

274. We proceed next to the investigation of the accelerations of the various points of a rigid body having **plane motion.** The motion is determined by that of a plane section of the body parallel to the plane of motion, and this consists in the rolling of the body centrode over the space centrode (Art. 22).

During any element of time dt, every point P of the plane section rotates with angular velocity ω about the instantaneous centre of rotation C which is the point of contact of the two centrodes. During the next element of time dt, the angular velocity is $\omega + d\omega$, and the centre of rotation has changed to the infinitely near point C_1 on the space centrode, which has now become the point of contact of the two centrodes. The acceleration of a point P at the distance r from C evidently depends on this distance r, the angular velocity ω, the angular acceleration $\alpha = d\omega/dt$, and the element $CC_1 = d\sigma$ of the space centrode. This element divided by dt may be regarded as a velocity, $u = d\sigma/dt$, viz. the velocity with which the instantaneous centre changes its position. We may call it the *velocity of rolling* of the body centrode. The change in the state of motion during two consecutive elements of time depends on α and u.

275. The relation of the velocity of rolling u to the angular velocity ω depends on the relative curvature of the centrodes c, c'.

To fix the ideas imagine these curves to lie on the same side of their common tangent; let $d\alpha$, $d\alpha'$ be their angles of contingence, and let ρ, ρ' be their radii of curvature (Fig. 70).

The rotation about C brings the second element of c' to coincidence with the second element of c. The angle $d\theta$ of this rotation is therefore equal to the difference of the angles of contingence of the two curves, *i.e.*

$$d\theta = d\alpha' - d\alpha.$$

This angle is therefore called the *angle of relative contingence;* the quotient $d\theta/d\sigma=(d\alpha'-d\alpha)/d\sigma$, where $d\sigma=CC_1$, is called the *relative curvature*, and the reciprocal value, $d\sigma/d\theta$, is the *radius of relative curvature.*

Now $\omega=d\theta/dt$, $u=d\sigma/dt$; hence

$$\frac{\omega}{u}=\frac{d\theta}{d\sigma}=\frac{d\alpha'-d\alpha}{d\sigma},$$

or as $d\alpha/d\sigma=1/\rho$, $d\alpha'/d\sigma=1/\rho'$,

$$\frac{\omega}{u}=\frac{d\theta}{d\sigma}=\frac{1}{\rho'}-\frac{1}{\rho}; \tag{3}$$

i.e. *the ratio of the angular velocity to the velocity of rolling is equal to the relative curvature of the centrodes.*

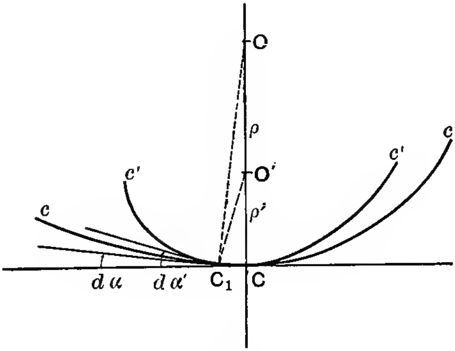

Fig. 70.

When $\rho<\rho'$, that is when $d\alpha>d\alpha'$, the relative curvature is negative. When the centrodes lie on opposite sides of the common tangent we should find in absolute value $d\theta=d\alpha'+d\alpha$. But taking into account the sense of the angles $d\alpha$, $d\alpha'$ we still have $d\theta=d\alpha'-d\alpha$. The formula (3) holds, therefore, generally if the radius of curvature ρ of c be taken as positive or negative according as it lies on the same side of the common tangent with the radius of curvature ρ' of c', or on the opposite side.

276. To determine the *components of the acceleration of any point* P *of the body,* it will be convenient to imagine the angular velocities represented by their rotors: the velocity ω about C by a line of length ω erected at C at right angles to the plane of motion, on that side of this plane from which the rotation appears counter-clockwise; similarly the angular velocity $\omega+d\omega$ by a parallel line of length $\omega+d\omega$ erected at C_1.

The rotor $\omega+d\omega$ through C_1 can be replaced by a parallel rotor of the same magnitude and sense through C, in combination with a rotor-couple whose moment is $(\omega+d\omega)\cdot CC_1=\omega d\sigma$ (see Arts. 255, 256). This couple being equivalent to a vector $\omega d\sigma$ at right angles to the plane of the couple produces an infinitesimal velocity of translation.

Thus the body, during the first element of time dt, rotates about the axis through C with angular velocity ω; and during the second element of time dt, it can be regarded as having the angular velocity $\omega+d\omega$ about the same axis, and at the same time a velocity of translation $\omega d\sigma$ at right angles to the tangent at C. The change in the state of motion consists, therefore, in the angular acceleration $d\omega/dt=\alpha$ and in the linear acceleration $\omega d\sigma/dt=\omega u$, the former being due to the change in the magnitude of the acceleration, the latter to the change in the position of the axis of rotation.

While the acceleration of translation ωu is the same for all points of the figure, the angular acceleration α produces in every point P (Fig. 71) a linear acceleration proportional to its distance $r=CP$ from the centre C, just as in the case of rotation about a fixed axis (Art. 273). Resolving this acceleration into its tangential and normal components we have for the acceleration of P the following three components: αr at right angles to CP, ω^2r along PC, and ωu at right angles to CC_1.

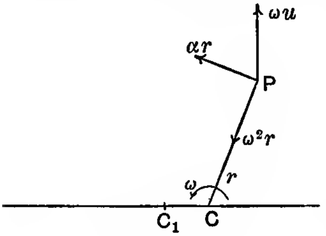

Fig. 71.

277. Another important method for finding the components of the acceleration of any point P of the body consists in resolving (according to Art. 254) the rotor $\omega+d\omega$ through C_1 into two parallel rotors, ω through C, and $d\omega$ through a point H (Fig. 72) on the tangent CC_1 whose distance $CH=h$ from C is given by the relations

$$\frac{CC_1}{d\omega}=\frac{C_1H}{\omega}=\frac{CH}{\omega+d\omega}.$$

Putting again $CC_1=d\sigma$, $d\sigma/dt=u$, $d\omega/dt=\alpha$, we find for the distance $CH=h$:

$$h=\frac{\omega u}{\alpha}. \tag{4}$$

The body can therefore be regarded as having, during the second element of time dt, the same angular velocity ω about the same axis through C as during the first element of time, but in addition an angular velocity $d\omega$ about a parallel axis through H. As the magnitude of the angular velocity about C does not change, the rotation about C produces at any point P (Fig. 72) only a normal acceleration ω^2r towards C, but no tangential acceleration. The infinitesimal angular velocity $d\omega$ about H, on the other hand, produces only a tangential acceleration $\alpha r'$, perpendicular to $HP=r'$.

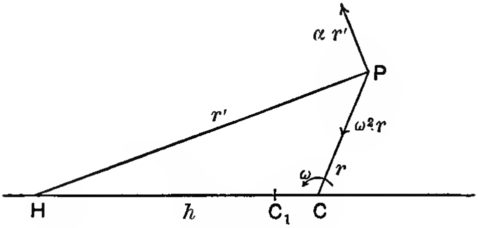

Fig. 72.

The acceleration of any point P can therefore be resolved into two components, one ω^2r directed towards the centre of rotation C and proportional to the distance r from this centre, the other $\alpha r'$ perpendicular and proportional to the distance r' of P from a point H on the tangent at C, such that $CH = \omega u/\alpha$.

The point H may be called the **centre of angular acceleration.**

278. The resolution of the acceleration given in the last article enables us to show the existence, at any time t, of a point having at this instant no acceleration. This point is called the instantaneous **centre of acceleration**; we shall denote it by the letter I, and its distances from C and H, respectively, by r_0 and r_0'.

For a point of acceleration zero the components $\alpha r'$ and $\omega^2 r$ must be equal and opposite. Now it is evident that these components fall into the same straight line only for points whose radii vectores r, r' are at right angles. The centre I must therefore lie on the circle described over CH as diameter (Fig. 73). In addition to this the radii vectores of I must fulfil the condition

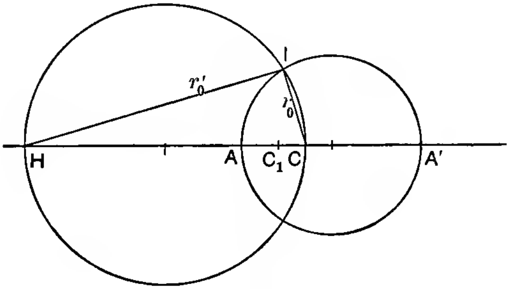

Fig. 73.

$$\omega^2 r_0 = \alpha r_0'. \tag{5}$$

The locus of all points for which at any instant the ratio r/r' is constant and equal to α/ω^2 is a circle whose centre lies on CH and whose intersections A, A' with CH divide this distance internally and externally in the ratio α/ω^2.

The two circles intersect in two points; but only for one of these have the components $\omega^2 r$ and $\alpha r'$ opposite sense. There exists therefore only one centre of acceleration I, and its radii vectores satisfy the conditions

$$\omega^2 r_0 = \alpha r_0', \quad r_0^2 + r_0'^2 = h^2 = \frac{\omega^2 u^2}{\alpha^2}. \tag{6}$$

279. The appropriateness of the name centre of acceleration for the point I appears in particular when the acceleration of *any* point P is referred to this point I. For it can be shown that, if p be the distance of P from I, *the acceleration of* P *can be resolved into two components, one* ω^2p *along* PI, *the other* αp *at right angles to* IP (Fig. 74), similarly as in the case of rotation about a fixed axis (see Art. 273).

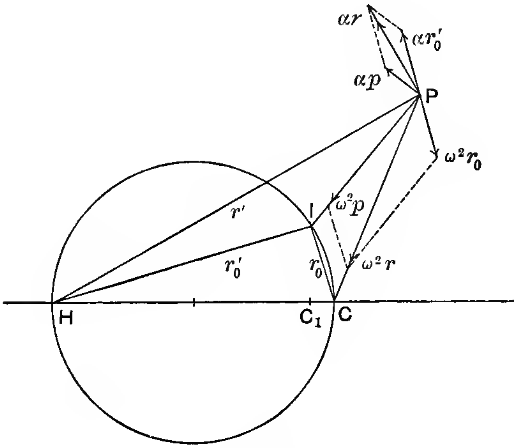

Fig. 74.

To prove this we resolve the component $\omega^2 r$ of the acceleration of P along PI and parallel to IC; it appears from the figure that these components are $\omega^2 p$ and $\omega^2 r_0$. The other component $\alpha r'$ of the acceleration of P is due to the infinitesimal angular velocity $d\omega$ about H. Replacing this $d\omega$ about H by an equal angular velocity $d\omega$ about I in combination with the infinitesimal velocity of translation $r_0' d\omega$ at right angles to

HI, we obtain, in the place of $\alpha r'$, the components αp at right angles to IP and $\alpha r_0'$ perpendicular to r_0'.

As of the four components $\omega^2 p$, αp, $\omega^2 r_0$, $\alpha r_0'$ the last two are, by (6), equal and opposite, it follows that the acceleration of P has only the two components, $\omega^2 p$ along PI, and αp perpendicular to IP.

280. The total acceleration of any point P is therefore proportional to the distance p of this point from the centre of acceleration I, viz.

$$j = p\sqrt{\alpha^2 + \omega^4}; \tag{7}$$

and the angle ψ it makes with this distance IP, being given by the relation

$$\tan\psi = \frac{\alpha}{\omega^2}, \tag{8}$$

is the same for all points. By (5), this angle ψ is equal to the angle CHI.

All points on a circle described about I as centre have accelerations of equal magnitude but of different directions. All points on a straight line drawn through I have accelerations that are parallel but differ in magnitude.

281. Returning to the resolution of the acceleration into three components $\omega^2 r$, αr, ωu, as given in Art. 276, let us take the common tangent of the centrodes as axis of x, their normal as axis of y (Fig. 75), and let x, y be the co-ordinates of any point P whose distance from C is $CP = r$.

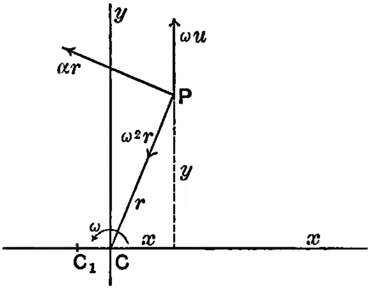

Fig. 75.

As the direction cosines of $\omega^2 r$, αr, ωu are respectively $-x/r$, $-y/r$; $-y/r$, x/r; 0, 1, we have for the components of the acceleration j parallel to the axes:

$$j_x = -\omega^2 x - \alpha y, \qquad \qquad (9)$$
$$j_y = -\omega^2 y + \alpha x + \omega u.$$

The co-ordinates x_0, y_0 of the centre of acceleration I must fulfil the conditions

$$\omega^2 x_0 + \alpha y_0 = 0, \quad \alpha x_0 - \omega^2 y_0 + \omega u = 0, \qquad (10)$$

whence
$$x_0 = \frac{\omega \alpha u}{\alpha^2 + \omega^4}, \quad y_0 = -\frac{\omega^3 u}{\alpha^2 + \omega^4}. \qquad (11)$$

The equations (10) evidently represent the two lines CI and HI.

282. Let $\xi = x - x_0$, $\eta = y - y_0$ be the co-ordinates of P with respect to parallel axes through I; then, combining (10) and (9), we find

$$j_x = -\omega^2 \xi - \alpha \eta, \; j_y = -\omega^2 \eta + \alpha \xi. \qquad (12)$$

These expressions show that the total acceleration j of P is

$$j = \sqrt{\alpha^2(\xi^2 + \eta^2) + \omega^4(\xi^2 + \eta^2)} = p\sqrt{\alpha^2 + \omega^4},$$

since $\sqrt{\xi^2 + \eta^2} = p = IP$, as in Art. 280.

283. The tangential and normal components of the acceleration j are readily obtained from Fig. 74, as follows:

$$j_t = \alpha r + \omega u \cdot \frac{x}{r}, \quad j_n = \omega^2 r - \omega u \cdot \frac{y}{r}. \qquad (13)$$

The loci of the points having only normal, and only tangential, acceleration at any moment are therefore the circles:

$$\alpha(x^2 + y^2) + \omega u x = 0, \quad \omega(x^2 + y^2) - u y = 0. \qquad (14)$$

284. Exercises.

(1) A wheel of radius a rolls on a straight track. Find the centre of angular acceleration H, (a) when the velocity v with which the axis of the wheel moves along the track is constant; (b) when v is uniformly accelerated as when the wheel rolls down an inclined plane; (c) when v is uniformly retarded, as in rolling up an inclined plane.

(2) Show that ωu is the total acceleration of the instantaneous centre C.

(3) Show that the points of the semi-circle described over CH as diameter and containing I have no tangential acceleration, and that for points without the circle about CH the velocity is increasing while for points within it is decreasing.

(4) Find the locus of the points of equal tangential acceleration.

(5) Show that the locus of the points having no normal acceleration at a given instant is a circle touching the common tangent of the centrodes at C and passing through I. This circle is called the *circle of inflexions;* give the reason for this name.

(6) Find the locus of the points having equal normal acceleration.

(7) Show that the diameter of the circle of inflexions is equal to the radius of relative curvature of the centrodes.

(8) Determine the locus of the points whose acceleration at any instant is parallel (a) to the common normal, (b) to the common tangent of the centrodes.

IV. *Solid Kinematics.*

I. MOTION OF A POINT IN A TWISTED CURVE.

285. We have so far considered only those cases of motion where the path of the point is a plane curve. In the most general case when the path is a so-called twisted or tortuous curve we may refer it to three rectangular axes and resolve the velocity v as well as the acceleration j each into three rectangular components parallel to these axes:

$$v_x = v\cos\alpha = \frac{dx}{dt}, \qquad j_x = j\cos\lambda = \frac{dv_x}{dt} = \frac{d^2x}{dt^2},$$

$$v_y = v\cos\beta = \frac{dy}{dt}, \qquad j_y = j\cos\mu = \frac{dv_y}{dt} = \frac{d^2y}{dt^2},$$

$$v_z = v\cos\gamma = \frac{dz}{dt}, \qquad j_z = j\cos\nu = \frac{dv_z}{dt} = \frac{d^2z}{dt^2},$$

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2};$$

$$j = \sqrt{j_x^2 + j_y^2 + j_z^2} = \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2 + \left(\frac{d^2z}{dt^2}\right)^2}.$$

286. As polar co-ordinates of the point P we take the radius vector $OP = r$, the colatitude $xOP = \theta$, and the longitude $yOQ = \phi$, Q being the projection of P on the plane yOz (Fig. 76).

The velocity v can be resolved into three rectangular components: v_r along r, v_θ at right angles to r in the plane xOP of the angle θ, and v_ϕ at right angles to this plane. To find their values we take the element $PP' = ds$ of the curve described by the point P as diagonal of an infinitesimal parallelopiped having its edges in those three rectangular directions. The three edges concurring in P are evidently dr, $rd\theta$, $r\sin\theta\, d\phi$; hence the components of the velocity are

$$v_r = \frac{dr}{dt}, \quad v_\theta = r\frac{d\theta}{dt}, \quad v_\phi = r\sin\theta\frac{d\phi}{dt}.$$

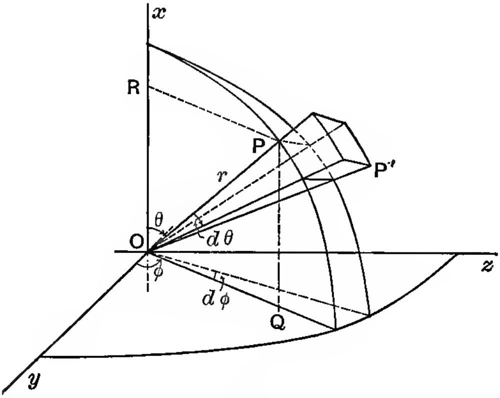

Fig. 76.

287. The components of the acceleration j in polar co-ordinates are readily obtained by considering that the accelerations of the point P in the direction at right angles to Ox in the plane xOP and in the direction at right angles to this plane are the same as the accelerations of the point Q (Fig. 76); they are therefore, by Art. 161, (6), since $RP = OQ = r\sin\theta$,

$$\frac{d^2(r\sin\theta)}{dt^2} - r\sin\theta\left(\frac{d\phi}{dt}\right)^2$$

in the direction RP, and

$$\frac{1}{r\sin\theta}\frac{d}{dt}\left(r^2\sin^2\theta\frac{d\phi}{dt}\right)$$

at right angles to the plane of the angle θ. The component of j parallel to the axis Oz is, of course, $d^2(r\cos\theta)/dt^2$. Resolving these three components parallel to the three rectangular directions along r, at right angles to r in the plane xOP, and at right angles to this plane, and collecting the terms, we obtain :

$$j_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 - r\sin^2\theta\left(\frac{d\phi}{dt}\right)^2,$$

$$j_\theta = r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} - r\sin\theta\cos\theta\left(\frac{d\phi}{dt}\right)^2,$$

$$j_\phi = r\sin\theta\frac{d^2\phi}{dt^2} + 2\sin\theta\frac{dr}{dt}\frac{d\phi}{dt} + 2r\cos\theta\frac{d\theta}{dt}\frac{d\phi}{dt}.$$

288. It is to be noticed that the resolution of the acceleration j into a tangential component j_t and a normal component j_n,

$$j_t = \frac{dv}{dt}, \quad j_n = \frac{v^2}{\rho},$$

given in Art. 159, holds for twisted curves as well as for plane curves, provided the normal be understood to mean the principal normal of the curve, and ρ the radius of absolute curvature at P. For it follows from the definition given in Art. 155 that the acceleration lies in the plane of the tangent and principal normal at P, so that the component along the binormal is zero.

289. This can also be seen from the expressions for the components of j in Cartesian co-ordinates, $j_x = d^2x/dt^2$, $j_y = d^2y/dt^2$, $j = d^2z/dt^2$. For since $\frac{dx}{dt} = \frac{dx}{ds}\frac{ds}{dt}$, etc., we have

$$j_x = \frac{d^2x}{dt^2} = \frac{d^2s}{dt^2}\frac{dx}{ds} + \left(\frac{ds}{dt}\right)^2\frac{d^2x}{ds^2},$$

$$j_y = \frac{d^2y}{dt^2} = \frac{d^2s}{dt^2}\frac{dy}{ds} + \left(\frac{ds}{dt}\right)^2\frac{d^2y}{ds^2},$$

$$j_z = \frac{d^2z}{dt^2} = \frac{d^2s}{dt^2}\frac{dz}{ds} + \left(\frac{ds}{dt}\right)^2\frac{d^2z}{ds^2}.$$

Now, dx/ds, dy/ds, dz/ds are the direction cosines of the tangent of the curve, while $\rho\, d^2x/ds^2$, $\rho\, d^2y/ds^2$, $\rho\, d^2z/ds^2$ are the direction cosines of the principal normal. The formulæ show therefore that the acceleration j consists of two components, $\frac{d^2s}{dt^2}=\frac{dv}{dt}$ along the tangent, and $\frac{1}{\rho}\left(\frac{ds}{dt}\right)^2=\frac{v^2}{\rho}$ along the normal.

2. VELOCITIES IN THE RIGID BODY.

290. When the motion of a rigid body is a *translation*, all points of the body have at any instant equal and parallel velocities (Art. 250). The velocity $v=ds/dt$ of any one point can therefore be called the velocity of the body. The body can be subjected at a given instant to several velocities of translation, and the resultant velocity is found by the geometrical addition of the vectors representing the component velocities.

291. When a rigid body *rotates* at the time t about an instantaneous axis l, all its points (excepting those on the axis) describe infinitesimal arcs of circles of angle $d\theta$, and the angular velocity $\omega=d\theta/dt$ of any point of the body may be called the angular velocity of the body. This angular velocity can be represented geometrically by its rotor ω laid off on the axis l (Arts. 68, 69, 252).

As this rotor is proportional to the infinitesimal angle of rotation $d\theta$, the propositions proved in Arts. 62, 66, 67, 68, for the composition and resolution of infinitesimal rotations can be applied directly to angular velocities. The propositions referring to parallel axes have been discussed in Arts. 254–257.

292. If in Art. 62 we divide equation (1′) by dt^2 and divide the denominators of equation (2′) by dt, we obtain

$$\omega^2=\omega_1^2+\omega_2^2+2\,\omega_1\omega_2\cos(l_1l_2), \tag{1}$$

$$\frac{\sin(l_1l)}{\omega_2}=\frac{\sin(ll_2)}{\omega_1}=\frac{\sin(l_1l_2)}{\omega}. \tag{2}$$

The meaning of these equations can be stated as follows. Let a rigid body be subjected simultaneously to two angular velocities about intersecting axes, ω_1 about l_1 and ω_2 about l_2. Represent these angular velocities by their rotors ω_1, ω_2 laid off on the axes l_1, l_2 from their point of intersection O and construct their geometric sum ω; that is, form the diagonal of the parallelogram whose adjacent sides are ω_1, ω_2. Then ω is the rotor of the resulting angular velocity.

This proposition is known as the **parallelogram of angular velocities.**

It follows that the resultant of any number of simultaneous angular velocities whose axes all intersect in the same point is a single angular velocity whose rotor is found by geometrically adding the rotors of the components.

293. Conversely, an angular velocity ω about an axis l can always be replaced, in an infinite number of ways, by two (or more) angular velocities whose geometric sum is ω, about two (or more) axes passing through any point O of l and lying in the same plane with l.

Thus, for instance, the angular velocity ω about the instantaneous axis l can be resolved into three components ω_x, ω_y, ω_z about three rectangular axes Ox, Oy, Oz passing through any point O of l, and we have

$$\omega^2 = \omega_x^2 + \omega_y^2 + \omega_z^2. \tag{3}$$

The linear velocity v of any point P of a body rotating with angular velocity ω about the axis l can be expressed by means of the components ω_x, ω_y, ω_z of ω and the co-ordinates x, y, z of the point P. The component ω_x produces at P a velocity whose components along the axes Ox, Oy, Oz are 0, $-\omega_x z$, $\omega_x y$; similarly, ω_y gives the components $\omega_y z$, 0, $-\omega_y x$; and ω_z gives $-\omega_z y$, $\omega_z x$, 0. Hence, combining the terms that lie along the same axis, the components of the velocity v of the point P are

$$\frac{dx}{dt} = \omega_y z - \omega_z y,\ \frac{dy}{dt} = \omega_z x - \omega_x z,\ \frac{dz}{dt} = \omega_x y - \omega_y x. \tag{4}$$

294. If a rigid body be subjected at the time t to two simultaneous angular velocities ω_1, ω_2 about skew (or crossing, *i.e.* not intersecting and not parallel) axes l_1, l_2, or if it be subjected to an angular velocity ω about an axis l and a simultaneous linear velocity v not perpendicular to l, its state of motion during the time dt cannot be expressed by a single angular or linear velocity.

The body can be said to have in either case a *twist-*, or *screw-velocity*, *i.e.* an angular velocity ω about an axis l combined with a linear velocity v_0 parallel to this axis.

To prove this in the latter of the two cases it is only necessary to resolve v into a component v_0 parallel to l and a component v' perpendicular to l. The latter, being equivalent to a rotor couple $(\omega, -\omega)$ of moment $v' = p\omega$ (see Art. 256), combines with the given angular velocity ω about l into an angular velocity ω about a parallel axis l' at the distance $p = v'/\omega$ from l. The combination of the angular velocity ω about l with the simultaneous oblique linear velocity v is therefore equivalent to the angular velocity ω about l' with the simultaneous linear velocity v_0 parallel to l'.

295. When the rigid body has two simultaneous angular velocities ω_1, ω_2 about skew axes l_1, l_2, the reduction is best made by replacing ω_2 about l_2 by an equal angular velocity ω_2 about a parallel axis l' intersecting l_1, in combination with a linear velocity $v = p\omega_2$ perpendicular to the plane of l_2 and l' (Art. 257). The angular velocities ω_1 about l_1 and ω_2 about l' combine (by Art. 292) into a singular angular velocity whose rotor is the geometric sum of ω_1 and ω_2. The case is therefore reduced to the preceding one.

296. It follows from the preceding articles that any number of simultaneous linear and angular velocities can always be combined into a single twist-velocity about the central axis.

3. ACCELERATIONS IN THE RIGID BODY.

297. The accelerations of the points of a rigid body are found by comparing the velocities of these points during two successive elements of time.

If the motion of the rigid body be a pure **translation**, all points of the body describe equal and parallel curves. The accelerations of all points being equal and parallel (Art. 272), the acceleration j of any one point of the body can be spoken of as *the acceleration of the body.* It can be resolved into a *tangential component* j_t along the tangent to the path of any point and a *normal component* j_n along the normal to the path, and we have, just as in Art. 159,

$$j_t = \frac{dv}{dt}, \quad j_n = \frac{v^2}{\rho},$$
$$j = \sqrt{j_t^2 + j_n^2} = \sqrt{\left(\frac{dv}{dt}\right)^2 + \frac{v^4}{\rho^2}}. \qquad (1)$$

298. If the motion of the rigid body be a pure **rotation** about the same axis l for at least two successive elements of time dt, all points describe arcs of circles whose centres lie on the fixed axis l. As shown in Art. 273, the acceleration j of any point P whose distance from l is r can be resolved into a *tangential component* j_t perpendicular to the plane (l, P) and a *normal component* j_n at right angles to the axis l; and we have (Art. 273)

$$j_t = \alpha r, \quad j_n = \omega^2 r,$$
$$j = \sqrt{j_t^2 + j_n^2} = r\sqrt{\alpha^2 + \omega^4}, \qquad (2)$$

where ω is the angular velocity and $\alpha = d\omega/dt$ the angular acceleration of the body.

The normal component j_n being always directed towards the axis of rotation l is sometimes called the **centripetal acceleration**.

299. If the motion of the rigid body consists in a rotation about an axis l during the first element of time and a rotation about an infinitely near *parallel* axis l' during the second element of time, we have the case of **plane motion** of a rigid body which has been treated in Arts. 274–284.

It remains to discuss the case of *intersecting axes*, which is of fundamental importance in the kinetics of the rigid body.

When the axes about which the body rotates in the successive elements of time intersect at a point O, this point remains fixed during the motion and may be called the *centre of rotation*. The motion of a rigid body with a fixed point may be called **spherical motion.**

The accelerations of the points of a body in spherical motion can be studied in a manner strictly analogous to that used in the case of plane motion (Arts. 274–284).

300. Let the body rotate during the first element of time dt with angular velocity ω about an axis l, and during the second element of time dt with angular velocity $\omega + d\omega$ about an axis l' intersecting l in the point O and making with l the infinitesimal angle $(l, l') = d\sigma$. The angular velocities can be represented by their rotors, ω along l, $\omega + d\omega$ along l' (Fig. 77).

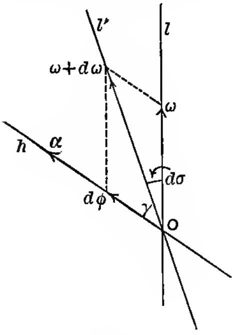

Fig. 77.

The rotor $\omega + d\omega$ along l' can be resolved into a rotor ω along l and an infinitesimal rotor $d\phi$ along an axis h that passes through O and lies in the plane (l, l'). The value of $d\phi$ and the angle $(l, h) = \gamma$ are given by the relations

$$\frac{\sin(l, l')}{d\phi} = \frac{\sin(l', h)}{\omega} = \frac{\sin(l, h)}{\omega + d\omega}, \qquad (3)$$

whence
$$\sin(l, h) \equiv \sin\gamma = \omega \frac{d\sigma}{d\phi}. \qquad (4)$$

Putting $d\sigma/dt = u$, $d\phi/dt = \alpha$, we may, similarly, as in Art. 274, call u the *velocity of rolling* of the cone of instantaneous axes and α the *angular acceleration*. With these notations

$$\sin\gamma = \frac{\omega u}{\alpha}. \tag{5}$$

301. The appropriateness of these names will appear by considering that the body can now be regarded as having, for two successive elements of time, the same angular velocity ω about the same axis l, modified during the second element of time by the additional infinitesimal angular velocity $d\phi$ about the axis h, which is called the *axis of angular acceleration*.

Thus the rotation about l produces only *centripetal* (and no tangential) *acceleration* which at unit distance from l is $=\omega^2$ and is directed at right angles to l towards l (see Art. 298), while the rotation about h gives at unit distance from h the infinitesimal velocity $d\phi$ at right angles to the planes through h and thus produces the *angular acceleration* $\alpha = d\phi/dt$, which may be represented by a vector α along h.

The projection of $d\phi$ on l is evidently $d\omega$ (see Fig. 77), so that

$$\cos\gamma = \frac{d\omega}{d\phi} = \frac{1}{\alpha}\frac{d\omega}{dt}. \tag{6}$$

Squaring and adding the equations (5) and (6), we find

$$\alpha = \sqrt{\left(\frac{d\omega}{dt}\right)^2 + \omega^2 u^2}. \tag{7}$$

302. These results are further illustrated by another resolution analogous to that of Art. 276.

Imagine the body subjected, during the second element of time, to the equal and opposite angular velocities $\omega + d\omega$ and $-(\omega + d\omega)$ about l (Fig. 78); then combine $\omega + d\omega$ about l' with $-(\omega + d\omega)$ about l into the infinitesimal angular velocity $(\omega + d\omega)\sin d\sigma = \omega d\sigma$ about an axis n through O at right angles to l in the plane (l, l'). This is equivalent to resolving the rotor $\omega+d\omega$ along l' into the rotors $\omega+d\omega$ along l and $\omega d\sigma$ along n.

The body can now be regarded as rotating during both elements of time about the axis l, viz. during the first element with angular velocity ω, during the second with angular velocity $\omega+d\omega$, and in addition to that during the second element about the axis n with the infinitesimal angular velocity $\omega d\sigma$.

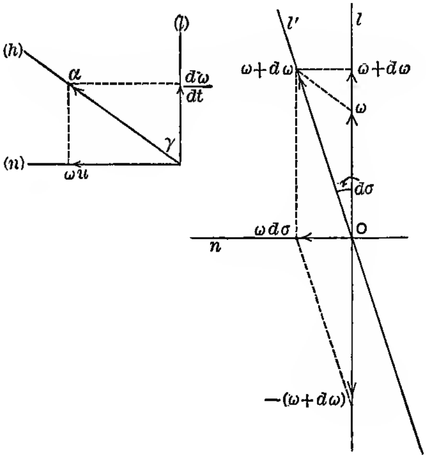

Fig. 78.

The rotation about l (Art. 298) produces, for points at unit distance from l, a *centripetal acceleration* ω^2 perpendicular to l and a *tangential acceleration* $d\omega/dt$ which may be represented by a rotor $d\omega/dt$ along l. The rotation about n gives to points at unit distance from n an infinitesimal velocity $\omega d\sigma$ at right angles to the planes through n and thus produces an acceleration $\omega d\sigma/dt=\omega u$ which may be represented by a rotor along n. The rotors $d\omega/dt$ along l and ωu along n being at right angles to each other (see Fig. 78), combine to form the *angular acceleration*

$$\alpha=\sqrt{\left(\frac{d\omega}{dt}\right)^2+\omega^2u^2}.$$

It is apparent that the component $d\omega/dt$ of α has the effect of changing the magnitude of ω by the amount $d\omega$, without affecting the direction of the axis, while the effect of the component ωu is to incline the axis l by the angle $d\sigma$.

303. To obtain analytical expressions for the components of the acceleration of any point P of a rigid body in spherical motion, let us take the centre of rotation O as origin of a system of fixed rectangular axes. Let x, y, z be the co-ordinates of P; α, β, γ the direction cosines of the instantaneous axis l; and λ, μ, ν those of the perpendicular $PQ=r$ let fall from P on this axis l.

The total acceleration of P is composed of the centripetal acceleration $\omega^2 r$, which is directed along PQ, and the component arising from the angular acceleration α (Art. 301).

The components of $\omega^2 r$ along the axes of x, y, z are $\lambda\omega^2 r$, $\mu\omega^2 r$, $\nu\omega^2 r$. Projecting the closed polygon $OQPO$ on each of the axes, we find

$$\alpha \cdot OQ=\lambda r+x, \quad \beta \cdot OQ=\mu r+y, \quad \gamma \cdot OQ=\nu r+z;$$

or, since OQ is the projection of OP on l, i.e. $OQ=\alpha x+\beta y+\gamma z$,

$$\begin{aligned}\lambda r&=\alpha(\alpha x+\beta y+\gamma z)-x,\\ \mu r&=\beta(\alpha x+\beta y+\gamma z)-y,\\ \nu r&=\gamma(\alpha x+\beta y+\gamma z)-z.\end{aligned}$$

Multiplying these equations by ω^2 and putting $\alpha\omega=\omega_x$, $\beta\omega=\omega_y$, $\gamma\omega=\omega_z$, we find for the components of the centripetal acceleration of the point (x, y, z):

$$\begin{aligned}\lambda\omega^2 r&=\omega_x(\omega_x x+\omega_y y+\omega_z z)-\omega^2 x,\\ \mu\omega^2 r&=\omega_y(\omega_x x+\omega_y y+\omega_z z)-\omega^2 y,\\ \nu\omega^2 r&=\omega_z(\omega_x x+\omega_y y+\omega_z z)-\omega^2 z.\end{aligned} \tag{8}$$

The angular acceleration $\alpha=d\phi/dt$ (Art. 301) has for its components along the axes of x, y, z

$$\alpha_x=\frac{d\omega_x}{dt}, \quad \alpha_y=\frac{d\omega_y}{dt}, \quad \alpha_z=\frac{d\omega_z}{dt}.$$

The component α_x produces an infinitesimal angular velocity $\alpha_x dt$ about the axis Ox; and hence gives to P the infinitesimal velocities 0, $-\alpha_x z dt$, $\alpha_x y dt$ along the axes Ox, Oy, Oz (see Art. 293); similarly, $\alpha_y dt$ produces the velocities $\alpha_y z dt$, 0, $-\alpha_y x dt$, and $\alpha_z dt$ produces $-\alpha_z y dt$, $\alpha_z x dt$, 0. Collecting the terms parallel to each axis and dividing by dt, we find the components of the acceleration of P due to the angular acceleration α:

$$\alpha_y z - \alpha_z y, \quad \alpha_z x - \alpha_x z, \quad \alpha_x y - \alpha_y x. \tag{9}$$

Finally, combining the corresponding terms in (8) and (9) and remembering that $\alpha_x = d\omega_x/dt$, $\alpha_y = d\omega_y/dt$, $\alpha_z = d\omega_z/dt$, we find the following expressions for the components of the total acceleration j of the point P (x, y, z):

$$\begin{aligned} j_x &= \omega_x(\omega_x x + \omega_y y + \omega_z z) - \omega^2 x + \frac{d\omega_y}{dt} z - \frac{d\omega_z}{dt} y, \\ j_y &= \omega_y(\omega_x x + \omega_y y + \omega_z z) - \omega^2 y + \frac{d\omega_z}{dt} x - \frac{d\omega_x}{dt} z, \\ j_z &= \omega_z(\omega_x x + \omega_y y + \omega_z z) - \omega^2 z + \frac{d\omega_x}{dt} y - \frac{d\omega_y}{dt} x. \end{aligned} \tag{10}$$

304. The formulas (10) for the components of the acceleration of any point (x, y, z) of a body rotating about a fixed point O can also be derived by differentiating the expressions (4) in Art. 293, which represent the components of the velocity of such a point. It is only necessary, after the differentiation, to substitute for dx/dt, dy/dt, dz/dt their values from (4), Art. 293, and to remember that $\omega^2 = \omega_x^2 + \omega_y^2 + \omega_z^2$.

305. The complete study of the motion of a rigid body in the most general case, in particular the investigation of its accelerations, is beyond the scope of the present work.

In addition to the works previously referred to, the following works on kinematics may here be mentioned.

An elementary introduction to kinematics, without the use of the infinitesimal calculus, will be found in J. G. MACGREGOR, *An elementary* *treatise on kinematics and dynamics*, London, Macmillan, 1887. This may be supplemented by W. K. CLIFFORD, *Elements of dynamic*, part 1, Kinematic, *ib.*, 1878. For more advanced study see G. M. MINCHIN, *Uniplanar kinematics of solids and fluids*, Oxford, Clarendon Press, 1882; THOMSON and TAIT, *Natural philosophy*, new edition, part 1, *ib.*, 1879; W. SCHELL, *Theorie der Bewegung und der Kräfte*, vol. 1, 1879, Leipzig, Teubner; J. SOMOFF, *Theoretische Mechanik*, übersetzt von A. Ziwet, part 1, Kinematik, Leipzig, Teubner, 1878; E. BUDDE, *Allgemeine Mechanik der Punkte und starren Systeme*, Berlin, Reimer, 1890; H. RESAL, *Traité de cinématique pure*, Paris, Mallet-Bachelier, 1862; E. BOUR, *Cours de mécanique et machines*, part 1, *Cinématique*, 2d ed., Paris, Gauthier-Villars, 1887; E. COLLIGNON, *Traité de mécanique*, part 1, *Cinématique*, 3d ed., Paris, Hachette, 1885; E. VILLIÉ, *Traité de cinématique*, Paris, Gauthier-Villars, 1888.

ANSWERS.

Page 17.

(1) Join the point P to the instantaneous centre C; the direction of motion is perpendicular to CP.

(3) See Art. 29.

(4) See Art. 29. With O as origin and a parallel to l as axis of y, the fixed centrode is $(y^2-cx)^2=a^2(x^2+y^2)$, where a is the radius of the circle about O, and c the distance of O from l.

(5) $y^2=2a(x+\frac{1}{2}a)$.

(6) The fixed centrode is a circle passing through O', O''; the body centrode is a circle of twice the radius. The path of any point in the fixed plane is in general a limaçon of Pascal.

(8) Consider the initial and final positions of the point of intersection of l_0 and l_1.

Page 28.

(1) 24 miles; E. 33° S.

(3) $\sqrt{3}$.

(4) 10.7 miles; E. $14\frac{1}{2}$° S.

(6) $2a\cos(\alpha/2)$.

(7) (a) 120°; (b) 160°48′.6.

(8) $(\sqrt{3}-1)a$; $\sqrt{2}(\sqrt{3}-1)a$.

(10) Inclination to vertical: (a) 11°.3; (b) 21°.8; (c) 45°; (d) 67°.4.

(11) $10\frac{1}{4}$ ft.; $\alpha=247\frac{1}{2}°$.

(16) On P_0P_1, P_0P_2 construct the parallelogram $P_0P_1RP_2$ and draw P_1S parallel to QP_2, S being the intersection with the diagonal P_0R.

(18) Apply (16).

Page 39.

(1) For the angle θ of the resulting rotation we have $\sin(\theta/2) = \frac{1}{2}\sqrt{5/2}$; for the position of its axis l, $\sin(l_1l) = 2/\sqrt{5}$, $\sin(ll_2) = \sqrt{2/5}$.

(3) 22°.

Page 47.

(1) (*a*) 41 miles per hour; (*c*) 19.1, 8.2; (*d*) 10 h. 19 m.

(3) At 2 h. 22 m.; 203 miles from Detroit.

(4) (*a*) 5.9; (*b*) 40.6; (*c*) 44; (*d*) 35.25; (*e*) 1093.

(5) 15.

(7) 185,000 miles per second.

(8) (*a*) 1 h.; (*b*) 15 m.

(9) $30\frac{3}{4}°$.

(10) $37\frac{1}{2}$ miles per hour.

Page 53.

(1) $\frac{11}{15}$.

(2) 32.186.

(3) Nearly $\frac{1}{5}$ ft. per second per second.

(4) 0.0034.

Page 56.

(1) (*a*) 96.6; (*b*) 402.5; (*c*) 144.9.

(2) 0.275.

(4) $h = c\left[t + \frac{c}{g} - \sqrt{\frac{c}{g}\left(2t + \frac{c}{g}\right)}\right]$; an approximate value is

$$h = \frac{t^2gc}{2(c+gt)}.$$

For a direct numerical computation, the method of successive approximations may be used. Thus, neglecting the time

t_2 required by the sound, find the depth s approximately from $s=\frac{1}{2}gt^2$, with $t=4$; with this value of s find t_2; hence the time of fall t_1, with which correct s; etc. Result: $s = 70.4$ metres.

(5) (*a*) 4 min.; (*b*) 11/60; (*c*) 30 miles per h.; (*d*) after 3 m. 28 s.

(8) (*a*) 40,000 ft.; (*b*) ± 715.5 ft. per second; (*c*) 1 m. 40 s.; (*d*) 1600 ft. per second; (*e*) 1 m. 12.4 s. and 27.6 s.

(9) 80 ft. per second.

(10) (*a*) $t = h/v_0$; (*b*) $h - s = \frac{1}{2}gh^2/v_0^2$; (*c*) $v_0 = \sqrt{gh}$.

Page 60.

(1) (*a*) 26,000 ft. per second; (*b*) 13 m. $32\frac{1}{2}$ s.

(2) It represents a cycloid.

(4) $v_0^2R/(2gR - v_0^2)$. If $v_0^2 \geqq 2gR$, the particle will not fall back.

(5) Height $= R$; time of ascending $= \dfrac{1}{\sqrt{g}}\left[1 + \dfrac{2\pi}{3}\sqrt{R}\right] = 28$ m. 24 s.; time of falling back, by (1), $=13$ m. 32 s.; hence whole time $= 41$ m. 56 s.

(6) 7 miles per second.

Page 63.

(2) $v = 2602$ ft. per second; $t = 1$ h. 25 m. 8.5 s.

(3) $2s = R(e^{\mu t} + e^{-\mu t})$, or $s = R\cosh \mu t$.

Page 65.

(1) $\lim v = g/\mu$ for $\lim t = \infty$.

(3) $$v = \frac{\sqrt{gk}\, v_0 \cos\sqrt{gk}\, t - g\sin\sqrt{gk}\, t}{\sqrt{gk}\cos\sqrt{gk}\, t + kv_0\sin\sqrt{gk}\, t},$$

$$s = \frac{1}{k}\log\left(\cos\sqrt{gk}\, t + \sqrt{\frac{k}{g}}\, v_0 \sin\sqrt{gk}\, t\right) = \frac{1}{2k}\log\frac{g + kv_0^2}{g + kv^2}.$$

(4) Time of ascent $$T = \frac{1}{\sqrt{gk}}\tan^{-1}\sqrt{\frac{k}{g}}\, v_0;$$

height of ascent $$H = \frac{1}{2k}\log\left(1 + \frac{k}{g}v_0^2\right).$$

(5) Compare the height of ascent in Ex. (4) to the distance fallen through as obtained in (27), Art. 126. If v_1 be the velocity with which the particle returns to the starting point, we find

$$v_1 : v_0 = \sqrt{g} : \sqrt{g + kv_0^2}.$$

(6) $v = v_0 e^{-kt}$, $s = \frac{v_0}{k}(1 - e^{-kt})$, $v = v_0 - ks$.

(7) $v = \frac{g}{k}(1 - e^{-kt})$, $s = \frac{g}{k^2}(kt + e^{-kt} - 1) = \frac{1}{k^2}\left(g \log \frac{g}{g - kv} - kv\right)$.

Page 69.

(1) $\omega = \pi$ radians; $v = 18.8$ ft. per second.

(2) (*a*) $3\frac{1}{3}$; (*b*) 32.

(3) -0.157.

(4) 5.

(5) (*a*) 402.1; (*b*) 25.1 seconds.

Page 73.

(1) $r = v_0 t$, $\theta = \omega t$; hence $r = v_0 \theta / \omega$, a spiral of Archimedes.

(2) About the pole O describe a circle of radius a and find its intersection Q with the perpendicular to the radius vector OP drawn through O; then QP is the normal. Proof by Ex. (1).

(3) For the direction of v see Art. 31, Ex. (2). Resolving v into v_0 parallel to the track and v_1 along the tangent to the wheel, it appears that v bisects the angle between these components; hence $v = 2 v_0 \cos CAP$, where C is the centre of the wheel, and A its lowest point.

(5) For the ellipse, $r + r' =$ const.; hence $\frac{dr}{dt} = -\frac{dr'}{dt}$, *i.e.* the projections of the velocity on the radii vectores are equal.

(6) The components of the velocity along the radius vector and parallel to the focal axis are in the constant ratio e of the focal radius vector to the distance to the directrix. It follows that the tangent intersects the directrix in the same point as does the perpendicular to the radius vector through O.

(7) 40.

(10) $v_1 = 20$ ft. per second, nearly; angle $= 20\frac{1}{4}°$.

(11) The relative velocity of P_2 with respect to P_1 must always pass through P_1. The locus of Q is a circle.

(12) A cycloid.

(13) About 20″.

Page 79.

(4) $233\frac{1}{3}$; $24\frac{4}{9}$; $933\frac{1}{3}$ ft. per minute.

(5) 16.6 knots; 560 ft. per minute.

(6) 55°; 66°; $2\frac{2}{3}$ in.

(7) 0.174, 0.119, 0.146 of the stroke.

Page 85.

(2) By (2), Art. 159, $j_n = \rho \left(\frac{d\alpha}{dt}\right)^2$.

(3) By Art. 159, $j_n = j \sin\psi = v^2/\rho$; hence $v^2 = j \cdot \rho \sin\psi$.

(4) Since j is directed towards A, taking A as origin, we have $j_\theta = 0$, *i.e.* $r^2 \frac{d\theta}{dt} = \text{const.}$; comp. Art. 135.

(5) $\frac{d\alpha}{dt} = \omega = \text{const.}$, $r = \text{const.}$; hence, by (6), Art. 161, $j = j_r = -r\omega^2$.

(6) $j = r\omega^2$.

Page 86.

(1) (*a*) 1718 ft. below the point; (*b*) after 2 m. 52 s.; (*c*) 1891 ft.; (*d*) 21.2 miles an hour.

(4) 45°.

(7) Construct a circle having the given point as its highest point and touching (*a*) the straight line, (*b*) the circle.

Page 90.

(9) (*a*) 174 ft.; (*b*) in about 8 seconds; (*c*) 257 ft. per second, inclined at an angle of about 5° to the vertical.

(10) 227 ft. per second.

(11) $4^\circ\ 21'$ or $86^\circ\ 48'$.

(13) Let $OV = v_0$ be the given initial velocity. On the vertical through O lay off $OD = H = v_0^2/2g$; then the horizontal through D is the directrix. Double the angle DOV, making $\measuredangle VOF = \measuredangle DOV$, and lay off $OF = OD = H$; then F is the focus.

Page 96.

(3) (*a*) 0, -4.985; (*b*) -2.22, -3.49; (*c*) -3.14, 0.

Page 101.

(1) $$x = 5 \cos\frac{2\pi}{10}t + 3\cos\left(\frac{2\pi}{10}t + 30^\circ\right) + 4\cos\left(\frac{2\pi}{10}t + 60^\circ\right)$$
$$= 10.805 \cos\left(\frac{\pi}{5}t + 27\tfrac{1}{8}^\circ\right).$$

(3) (*a*) $x = 2a\cos\omega t$; (*b*) $x = 0$.

(4) $x_1 = 5\sqrt{2}(1 - \sqrt{3})\cos\pi t$, $x_2 = 10\sqrt{2}\cos(\pi t + 30^\circ)$.

Page 120.

(1) 86138.7; 0.996976.

(2) 3.2545 ft.

(3) 32.60.

(4) 20 ft.

(5) 980.36.

(8) The pendulum should be shortened by 1/240 of its length.

(9) It will lose 67 seconds per day.

(11) About 2900 ft.

Page 123.

(3) 1.0039.

(5) As $\theta = \theta_0 = \pi$ when $v = v_0 = 0$, we have instead of (33), Art. 216, $\frac{1}{2}v^2 = 2gl\cos^2\frac{\theta}{2}$. Substituting $v = -l\frac{d\theta}{dt}$ and integrating, we find $\sqrt{\frac{g}{l}}\,t = \log\tan\left(\frac{\pi}{4} + \frac{\theta}{4}\right) + C$, which shows that the moving point approaches the highest point of the circle asymptotically.

Page 132.

(2) Let $x^2+y^2=a^2$ be the circle, the acceleration being parallel to the axis of y; then $j=-\dfrac{a^2v_1^2}{y^3}$, where v_1 is the x-component of the initial velocity.

(3) $f(r)=v_0^2/a$.

(4) Let $j=-\mu^2r$ be the acceleration; x_1, y_1 the initial position; v_1, v_2 the components of the initial velocity; then the path is the hyperbola

$$(v_2^2-\mu^2y_1^2)x^2+2(\mu^2x_1y_1-v_1v_2)xy+(v_1^2-\mu^2x_1^2)y^2=(v_2x_1-v_1y_1)^2.$$

(6) $a=\dfrac{\mu}{\epsilon^2}$, $\quad b=\dfrac{v_0r_0\sin\psi_0}{\epsilon}$, $\quad \tan\alpha=-\dfrac{v_0^2\sin 2\psi_0}{\epsilon^2+v_0^2\cos 2\psi_0}$, where $\epsilon^2=\dfrac{2\mu}{r_0}-v_0^2$.

Page 136.

(2) Find first the relative velocity of A' with respect to A, whence ω is obtained; determine the distances CA, CA'.

(4) $v_B=c\cot\phi$, $\quad v_{O'}=\frac{1}{2}c\csc\phi$, where $-c$ is the velocity of A.

Page 159.

(1) (*a*) At infinity; (*b*) at the distance $g\sin\theta\cdot t^2$ in front of C_1; (*c*) at the same distance back of C_1.

(3) ω^2r and $\alpha r'$ must fall into the normal C_1P if $j_t=0$; hence r must be perpendicular to r'. The component ω^2r is always normal; the tangential component is therefore $\alpha r'\cos HPC$; this is positive for points without the circle and negative for points within.

(4) The limaçon $r=h\cos\theta+\dfrac{j_n}{\alpha}$.

(5) The points of the circle of inflexions are points of inflexion in their respective paths. For points within the circle the path is convex, as seen from C_1, since their normal acceleration is negative; points without the circle have concave paths.

(6) A limaçon.

(8) (*a*) The line CI; (*b*) the line HI.

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